Solutions for January

479. Let x, y, z be positive integer for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

and the greatest common divisor of x and z is 1. Prove that $x+y$, $x-z$ and $y-z$ are all perfect squares. Give two examples of triples (x,y,z) that satisfy these conditions.

Solution 1. [G. Ghosn] Since $(1/y) = (x-z)/(xz)$ and gcd $(x, x-z) =$ gcd $(z, x-z) = 1$, the fractions on both sides of the equation are in lowest terms, and so $x - z = 1$ and $xz = y$. Hence $x + y = x(1+z) = x^2$ and $y - z = z(x-1) = z^2$.

Solution 2. Since $z(x+y) = xy$ and the greatest common divisor of x and z is 1, x, being a divisor of $z(x+y)$ must be a divisor of $x+y$ and so of y. Let $y = ux$ for some positive integer u. Then $z(1+u) = ux$. Since u and $1+u$ have greatest common divisor 1, u must divide z and $1+u$ must divide x, Hence $z = uv$ and $x = (1+u)w$, for some positive integers v and w. Therefore $uv(1+u) = u(1+u)w$, whence $v = w$.

Therefore $(x,y,z) = ((1+u)v, u(1+u)v, uv)$. Since x and z have greatest common divisor 1, $v = 1$ and $(x,y,z) = (1+u, u(1+u), u)$. This satisfies the given equation as well as $x + y = (1+u)^2 = x^2$, $x - z = 1$ and $y - z = u^2 = z^2$. Particular examples are $(x,y,z) = (2,2,1), (3,6,2), (4,12,3), (5,20,4)$.

Solution 3. We have that $z(x+y) = xy$ and $x(y-z) = yz$. Since gcd $(x,z) = 1$, z and x both must divide y, so that $y = vz = wx$ for some positive integers v and w. Since $z(1+w)x = xvz$, $1 + w = v$ and gcd $(v,w) = 1$. Since $wx = vz$,,we must have that $x = v$ and $z = w$ and $y = vw$. This satisfies the equation as well as $x + y = v^2$, $x - z = 1$ and $y - z = w^2$.

Solution 4. [K. Huynh] Observe that $x > y$ and $z > y$. From the equation, we obtain that $xz + yz = xy$ whence $(x-z)(y-z) = z^2$. Since gcd $(x,z) = 1$, there is no prime that divides $x - z$ and z^2, so that gcd $(x - z, z^2) = 1$. Therefore $x - z = 1$, $y - z = z^2$, $y = z^2 + z$ and $x + y = (z+1)^2$.

480. Let a and b be positive real numbers for which $60^a = 3$ and $60^b = 5$. Without the use of a calculator or of logarithms, determine the value of

$$12^{\frac{1-a-b}{2(1-b)}} \; .$$

Solution 1. [V. Zhou]

$$\begin{aligned} 12^{\frac{1-a-b}{2(1-b)}} &= \left(\frac{60}{5}\right)^{\frac{1-a-b}{2(1-b)}} = 60^{(1-b)\cdot(\frac{1-a-b}{2(1-b)})} \\ &= \left(\frac{60}{60^{a+b}}\right)^{\frac{1}{2}} = \left(\frac{60}{60^a \cdot 60^b}\right)^{\frac{1}{2}} \\ &= \left(\frac{60}{3 \times 5}\right)^{\frac{1}{2}} = 2 \; . \end{aligned}$$

Solution 2. Since $60^b = 5$, $12^b = 5^{1-b}$ and $5 = 12^{b/(1-b)}$. Since $60^a = 3$, $2^2 5^a 12^a = 12$. Therefore

$$2^2 = 12^{1-a} 5^{-a} = 12^{1-a} 12^{-ab/(1-b)} = 12^{(1-a-b+ab-ab)/(1-b)} = 12^{(1-a-b)/(1-b)} \; .$$

Therefore $2 = 12^{(1-a-b)/2(1-b)}$.

Solution 3. [A. Guo; D. Shi] Since $a = \log_{60} 3$ and $b = \log_{60} 5$,

$$1 - (a+b) = 1 - \log_{60}(15) = \log_{60}(60/15) = \log_{60} 4 \; .$$

Also, $1 - b = 1 - \log_{60} 5 = \log_{60} 12$, so that

$$\frac{1-a-b}{1-b} = \frac{\log_{60} 4}{\log_{60} 12} = \log_{12} 4 = 2\log_{12} 2 \ .$$

Therefore

$$12^{\frac{1-a-b}{2(1-b)}} = 12^{\log_{12} 2} = 2 \ .$$

481. In a certain town of population $2n+1$, one knows those to whom one is known. For any set A of n citizens, there is some person among the other $n+1$ who knows everyone in A. Show that some citizen of the town knows all the others.

Solution 1. [K. Huynh] We prove that there is a set of $n+1$ people in the town, each of whom knows (and is known by) each of the rest. First, observe that for any set of k people, with $k \leq n$, there is a person not among them who knows them all. This follows by augmenting the set to n people and applying the condition of the problem.

Let p_1 be any person. There is a person, say p_2 who knows p_1. A person p_3 can be found who knows both p_1 and p_2, so that $\{p_1, p_2, p_3\}$ is a triplet each of whom knows the other two. Suppose, as an induction hypothesis, that $3 \leq k \leq n$, and $\{p_1, p_2, \cdots, p_k\}$ is a set of k people any pair of whom know each other. By the foregoing observation, there is another person p_{k+1} who knows them all. By induction, we can find a set $\{p_1, p_2, \cdots, p_{n+1}\}$, each pair of whom know each other.

Consider the remaining n people. There must be one among the p_i who knows all of these remaining people. This person p_i therefore knows everyone.

Solution 2. Let us suppose that the persons are numbered from 0 to $2n$ inclusive. The notation $(a : a_1, a_2, \cdots, a_k)$ will mean that a is knows and is known by each of $a_1, a_2, \cdots, a_k$. Begin with the set $\{1, 2, \cdots, n\}$; some person, say 0, knows everyone in this set, so that

$$(0 : 1, 2, 3, \cdots, n) \ .$$

If person 0, knows everyone else, then we are done. Otherwise, there is a person, say, $n+1$, not known to 0, so that everyone in the set $\{n+1, n+2, \cdots, 2n\}$, is known by a person in the first set, say 1, so that

$$(1 : 0, n+1, n+2, \cdots, 2n) \ .$$

Consider the set $\{0, 2, 3, \cdots, n\}$. If 1 knows everyone in this set, then 1 knows everyone and we are done. If 1 does not know everyone in this set, then there is someone else, say $n+1$, who does, so that

$$(n+1 : 0, 1, \cdots, n) \quad \text{and} \quad (0 : 1, 2, \cdots, n+1) \ .$$

If 0 knows everyone in the set $\{1, n+2, \cdots, 2n\}$, then 0 knows everyone; if $n+1$ knows everyone in this set, then $n+1$ knows everyone, and we are done. If not, then there is a person 2, say, who knows everyone in the set:

$$(2 : 0, 1, n+1, n+2, \cdots, 2n) \ .$$

Consider the set $\{0, 3, \cdots, n, n+1\}$. If 1 or 2 knows everyone in this set, then 1 or 2 knows everybody and we are done. Otherwise, there is a person, say $n+2$ who knows everyone in the set, so that

$$(n+2 : 0, 1, 2, \cdots, n+1) \quad \text{and} \quad (0 : 1, 2, \cdots, n+1, n+2) \ .$$

We can continue on in this way either until we find someone that knows everyone, or until we reach the ith stage for which

$$(i : 0, 1, 2, \cdots, i-1, n+1, \cdots, 2n) \quad \text{and} \quad (n+i : 0, 1, 2, \cdots, n, n+1, \cdots, n+i-1) \ .$$

If we get to the nth stage, then n and $2n$ each know everyone.

482. A trapezoid whose parallel sides have the lengths a and b is partitioned into two trapezoids of equal area by a line segment of length c parallel to these sides. Determine c as a function of a and b.

Solution. Let u be the distance between the segment of length a and that of length c, and v the distance between the segment of length c and that of length b. Then

$$\frac{u+v}{u} = \frac{b-a}{c-a} \ .$$

From the area condition, we have that

$$2\left(\frac{c+a}{2}\right)u = \left(\frac{b+a}{2}\right)(u+v) = \left(\frac{b^2-a^2}{2(c-a)}\right)u \ ,$$

whence $2(c^2 - a^2) = b^2 - a^2$ and $c^2 = \frac{1}{2}(a^2 + b^2)$. Therefore

$$c = \sqrt{\frac{a^2+b^2}{2}} \ .$$

483. Let A and B be two points on the circumference of a circle, and E be the midpoint of arc AB (either arc will do). Let P be any point on the minor arc EB and N the foot of the perpendicular from E to AP. Prove that $AN = NP + PB$.

Solution 1. Produce ANP to M so that $AN = NM$. Then $EM = AE = EB$. Hence $\angle EBM = \angle EMB$, so that

$$\angle PBM = \angle EBM - \angle EBP = \angle EMB - \angle EAP = \angle EMB - \angle EMA = \angle PMB \ .$$

Therefore $PB = PM$, so that

$$AN = NM = NP + PM = NP + PB \ .$$

Solution 2. [V. Zhou] Determine Q on AN so that $AQ = BP$. Then, also, $\angle EAQ = \angle EAP = \angle EPB$ and $AE = EB$, so that triangles AEQ and BEP are congruent. Hence $EQ = EP$ and so $QN = NP$. Therefore $AN = QN + AQ = NP + PB$.

Solution 3. [Y. Wang] Let O be the centre and r the radius of the circle. Let F and G be the respective midpoints of AP and AB. Then $FG \| BP$ and, since $\angle AFO = \angle AGO = 90^\circ$, the quadrilateral $AFGO$ is concyclic.

Let $\alpha = \angle AOF = \angle AGF$ and $\beta = \angle AOE = \angle BOE$. Then

$$\angle PAB = \angle FAG = \angle FOG = \angle FOE = \angle NEO = \beta - \alpha \ .$$

Also, $|FN| = |OE| \sin(\beta - \alpha) = r \sin(\beta - \alpha)$ and $|AF| = r \sin \alpha$. By the Law of Sines applied to triangle AFG,

$$\frac{|FG|}{\sin(\beta - \alpha)} = \frac{|AF|}{\sin \alpha} = r,$$

whence $|FG| = r \sin(\beta - \alpha) = |FN|$. Hence $AN = PF + FN = PN + 2FN = PN + 2FG = NP + PB$.

484. ABC is a triangle with $\angle A = 40^\circ$ and $\angle B = 60^\circ$. Let D and E be respective points of AB and AC for which $\angle DCB = 70^\circ$ and $\angle EBC = 40^\circ$. Furthermore, let F be the point of intersection of DC and EB. Prove that $AF \perp BC$.

Solution 1. [J. Schneider] Let AH be the altitude from A to BC. We apply the converse of Ceva's Theorem in the trigonometric form to show that the cevians AH, BE and CD concur.

$$\frac{\sin 30^\circ \sin 40^\circ \sin 10^\circ}{\sin 10^\circ \sin 20^\circ \sin 70^\circ} = \frac{\sin 30^\circ (2 \sin 20^\circ \cos 20^\circ)}{\sin 20^\circ \cos 20^\circ} = 2 \sin 30^\circ = 1 \ .$$

Hence AH, BE and CD concur, so that AH passes through F and the result follows.

Solution 2. [A. Siddhour] In triangle BCF, since $\angle CBF = 40^\circ$ and $\angle CBF = 40^\circ$, it follows that $\angle BFC = 70^\circ = \angle CBF$ and $BF = BC$. Hence $|BF| = a$ (using the standard convention for lengths of the sides of the triangle ABC). Assign coordinates:

$$B \sim (0,0), \quad C \sim (a,0), \quad A \sim (c \cos 60^\circ, c \sin 60^\circ), \quad F \sim (a \cos 40^\circ, a \sin 40^\circ \ .$$

By the Law of sines, we have that $c \sin 40^\circ = a \sin 80^\circ$, whence $c = 2a \cos 40^\circ$.

We have that

$$\begin{aligned} \overrightarrow{FA} \cdot \overrightarrow{BC} &= (c \cos 60^\circ - a \cos 40^\circ, c \sin 60^\circ - a \sin 60^\circ) \cdot (a, 0) \\ &= a(2a \cos 40^\circ \cos 60^\circ - a \cos 40^\circ = a \cos 40^\circ - a \cos 40^\circ = 0 \ , \end{aligned}$$

from which it follows that $AF \perp BC$.

Solution 3. [Y. Wang] The result will follow if one can show that $\angle FAC = 10^\circ$. Since $\angle FCA = \angle BCA - \angle DCB = 80^\circ - 70^\circ = 10^\circ$, it is enough to show that the perpendicular from F to AC bisects AC, *i.e.*, $2|CF| \cos \angle FCA = |AC|$.

Since $\angle FBC = 40^\circ$ and $\angle BCF = 70^\circ$, it follows that $\angle BFC = 70^\circ$ so that $|CF| = 2|BC| \cos 70^\circ$. Since $BC : AC = \sin \angle BAC : \sin \angle ABC = \sin 40^\circ : \sin 60^\circ$,

$$2|CF| \cos \angle FCA = 4|BC| \cos 70^\circ \cos 10^\circ = 4|AC| \sin 40^\circ \sin 20^\circ \sin 80^\circ / \sin 60^\circ \ .$$

For each angle θ,

$$\begin{aligned} 4 \sin \theta \sin(60^\circ + \theta) \sin(60^\circ - \theta) &= 2 \sin \theta [\cos 2\theta - \cos 120^\circ] \\ &= 2 \sin \theta \cos 2\theta + 2 \sin \theta \sin 30^\circ \\ &= \sin 3\theta - \sin \theta + \sin \theta = \sin 3\theta \ . \end{aligned}$$

When $\theta = 20^\circ$, this becomes $4 \sin 20^\circ \sin 40^\circ \sin 80^\circ = \sin 60^\circ$. so that $2|CF| \cos \angle FCA = |AC|$, as desired.

Solution 4. Since $\angle BFC = 70^\circ = \angle BCD$, $BF = BC$. Let $|BF| = |BC| = 1$, $|AF| = u$ and $|CF| = v$. Let $\angle BAF = \theta$, so that $\angle CAF = 40^\circ - \theta$. By the Sine law applied to triangles BFC and AFC,

$$\frac{\sin 40^\circ}{\sin 70^\circ} = v = \frac{u \sin(40^\circ - \theta)}{\sin 10^\circ} \ .$$

By the Sine Law applied to triangle ABF, $u = \sin 20^\circ / \sin \theta$. Hence

$$\frac{\sin 40^\circ}{\sin 70^\circ} = \frac{\sin 20^\circ \sin(40^\circ - \theta)}{\sin 10^\circ \sin \theta} \ ,$$

so that

$$\sin 10^\circ \sin 40^\circ \sin \theta = \sin 20^\circ \cos 20^\circ \sin(40^\circ - \theta) \ ,$$

whence

$$2 \sin 10^\circ \sin \theta = \sin(40^\circ - \theta) = \sin 40^\circ \cos \theta - \cos 40^\circ \sin \theta$$

and

$$\sin\theta(2\sin 10^\circ + \cos 40^\circ) = \cos\theta \sin 40^\circ \ .$$

Now

$$\begin{aligned} 2\sin 10^\circ + \cos 40^\circ &= \sin 10^\circ + (\sin 10^\circ + \sin 50^\circ) \\ &= \sin 10^\circ + 2\sin 30^\circ \cos 20^\circ = \sin 10^\circ + \sin 70^\circ \\ &= 2\sin 40^\circ \cos 30^\circ = \sqrt{3}\sin 40^\circ \ . \end{aligned}$$

Hence $\sqrt{3}\sin\theta = \cos\theta$, so that $\cot\theta = \sqrt{3}$. Hence $\theta = 30^\circ$ and the result follows.

Solution 5. [K. Huynh] Let a, b, c be the sides of triangle ABC according to convention. Since $\angle BFC = \angle FCB = 70^\circ$, $|BF| = |BC| = a$. Let the respective feet of the perpendiculars from A and F to BC be P and Q. Then $|BP| = c\cos 60^\circ = c/2$ and $|BQ| = a\cos 40^\circ$. From the Law of Sines, $a\sin 80^\circ = c\sin 40^\circ$, so that $c = 2a\cos 40^\circ$. Hence $BP = BQ$, and the result follows.

Solution 6. [G. Ghosn] Applying the Law of Sines to triangles BCE and BEA using their common side BE, we obtain that

$$\frac{|EC|}{|EA|} = \left(\frac{\sin 40^\circ}{\sin 80^\circ}\right)\left(\frac{\sin 40^\circ}{\sin 20^\circ}\right) = \frac{\sin^2 40^\circ}{\sin 20^\circ \sin 80^\circ} = \frac{2\cos 20^\circ \sin 40^\circ}{\sin 80^\circ} \ .$$

Similarly,

$$\frac{|DA|}{|DB|} = \frac{\sin 10^\circ \sin 60^\circ}{\sin 40^\circ \sin 70^\circ} \ .$$

By Ceva's therem

$$\begin{aligned} 1 &= \frac{|EC|}{|EA|}\frac{|DA|}{|DB|}\frac{|MB|}{|MC|} \\ &= \frac{2\cos 20^\circ \sin 40^\circ \sin 10^\circ \sin 60^\circ}{\sin 80^\circ \sin 40^\circ \sin 70^\circ}\frac{|MB|}{|MC|} \\ &= \frac{2\cos 80^\circ \sin 60^\circ}{\sin 80^\circ}\frac{|MB|}{|MC|} \ , \end{aligned}$$

whence we find that $|MB| : |MC| = \tan 80^\circ : \tan 60^\circ$.

Let AN be an altitude of triangle ABC, so that $|AN| = |NB|\tan 60^\circ = |CN|\tan 80^\circ$. Then $MB : MC = NB : NC$, so that $M = N$ and the desired result follows.

485. From the foot of each altitude of the triangle, perpendiculars are dropped to the other two sides. Prove that the six feet of these perpendiculars lie on a circle.

Solution 1. Let ABC be the triangle with altitudes AP, BQ and CR; let H be the orthocentre. Let $PU \perp AB$, $QV \perp BC$, $RW \perp CA$, $PX \perp CA$, $QY \perp AB$ and $RZ \perp BC$, where $U, Y \in AB$; $V, Z \in BC$; and $W, X \in CA$.

Consider triangles AQR and ABC. Since $ARHQ$ is concyclic (right angles at Q and R),

$$\angle ARQ = \angle AHQ = \angle BHP = 90^\circ - \angle HBP = 90^\circ - \angle QBC = \angle ACB \ .$$

Similarly, $\angle AQR = \angle ABC$. Thus, triangles AQR and ABC are similar, the similarity being implemented by a dilatation of centre A followed by a reflection about the bisector of angle BAC. Since QY and RW are altitudes of triangle AQR, triangle AYW is formed from triangle AQR as triangle AQR is formed from triangle ABC. Hence triangles AYW and AQR are similar by the combination of a dilatation with centre A and a reflection about the bisector of angle BAC.

Therefore, triangle AYW and ABC are directly similar and $YW \| BC$. Similarly triangles BZU and BCA as well as triangles CXV and CAB are similar and $ZU \| CA$ and $XV \| AB$. (We note that this means that $XWYUZV$ is a hexagon with opposite sides parallel, although this is not needed here.)

Since $PX \| HQ$ and $PU \| HR$, $AU : AR = AP : AH = AX : AQ$, so that there is a dilatation taking $U \rightarrow R$, $P \rightarrow H$ and $X \rightarrow Q$. Therefore $UX \| RQ$ and triangle AXU is similar to triangle AQR and to triangle ABC.

Consider quadrilateral $UZVX$.

$$\begin{aligned} \angle UZV + \angle UXV &= (180^\circ - \angle BZU) + (180^\circ - \angle AXU - \angle CXV) \\ &= (180^\circ - \angle ACB) + (180^\circ - \angle ABC - \angle BAC) = 180^\circ \ . \end{aligned}$$

Hence $UZVX$ is concyclic. Similarly, $VXWY$ and $WYUZ$ are concyclic.

Since triangles AYW and AXU are similar with $\angle AWY = \angle AUX$ and $\angle AYW = \angle AXU$, $XWYU$ is concyclic. Similarly, $YUZV$ and $ZVXW$ are concylclic. Hence $XWYUZV$ is a hexagon, any consecutive four vertices of which are concylcic, and so is itself concyclic.

Solution 2. [K. Huynh] Let a, b, c be the lengths of the sides and A, B, C the angles of the triangle ABC according to convention. Use the notation of Solution 1. We have that $|BU| = |BP| \cos B = (c \cos B) \cos B = c \cos^2 B$. Similarly, $|BZ| = a \cos^2 B$, $|AY| = c \cos^2 A$ and $|CV| = a \cos^2 C$. Therefore, $|BY| = c(1 - \cos^2 A) = c \sin^2 A$ and $|CV| = a(1 - \cos^2 C) = a \sin^2 C$.

Since $a \sin C = c \sin A$,

$$\begin{aligned} |BU||BY| &= (c \cos^2 B)(a \sin^2 A) = \cos^2 B (c \sin A)^2 \\ &= \cos^2 B (a \sin C)^2 = (a \cos^2 B)(a \sin^2 C) = |BZ||BV| \ . \end{aligned}$$

from which, by a power-of-the-point argument [give details!], we deduce that $YUZV$ is concyclic. Similarly, $ZVXW$ and $XWYU$ are concyclic.

Suppose that the circumcircle of $YUZV$ intersects AZ at L and the circumcircle of $ZVXW$ intersects AZ at M. Since $XWYU$ is concyclic, $|AY||AU| = |AW||AX|$. Therefore,

$$|AL||AZ| = |AY||AU| = |AW||AX| = |AM||AZ| \ .$$

Hence $L = M$. Thus, the circumcircles of $YUZV$ and $ZVXW$ share three noncollinear points, Z, V and $L = M$, and so must coincide. Similarly, each coincides with the circumcircle of $XWYU$ and the result follows.