Solutions for September

451. Let a and b be positive integers and let $u = a + b$ and $v = lcm\ (a, b)$. Prove that

$$gcd\ (u, v) = gcd\ (a, b)\ .$$

Solution 1. Suppose that $d|a$ and $d|b$. Then d divides any multiple of these two numbers and so divides $lcm(a, b) = v$. Also, $d|a + b$. Hence $d|gcd(u, v)$.

On the other hand, suppose that $d|u$ and $d|v$. Let $g = gcd(d, a)$ and $d = gh$. We have that

$$v = lcm(a, b) = a \cdot \frac{b}{gcd(a, b)}\ .$$

Since d divides v, h divides d and $\gcd(h, a) = 1$, it follows that

$$h \Big| \frac{b}{gcd(a, b)}\ .$$

Now $g|a + b$ and $g|a$, so g divides $b = (a + b) - b$. Also $h|a + b$ and $h|b$, so h also divides a. But, as $gcd(h, a) = 1$, $h = 1$. Hence $d|a$. Similarly, $d|b$. Hence the pairs (a, b) and (u, v) have the same divisors and the result follows.

Solution 2. Let d be the greatest common divisor of a and b, and write $a = da_1$ and $b = db_1$. The pair (a_1, b_1) is coprime. We have that $u = d(a_1 + b_1)$ and $v = d(a_1b_1)$. The greatest common divisor of u and v is equal to $d \cdot gcd(a_1 + b_1, a_1b_1)$.

Suppose, if possible, that there is a prime p that divides both $a_1 + b_1$ and a_1b_1. Then p must divide one of the factors a_1, b_1 of the product, say a_1. Then p must also divide $b_1 = (a_1 + b_1) - a_1$, which contradicts the coprimality of the pair (a_1, b_1). Hence $gcd(a_1 + b_1, a_1b_1) = 1$, and the result follows.

Solution 3. Let $\gcd(a, b) = \prod p^k$, where the product is taken over all primes dividing the left side and p^k is the largest power of the prime dividing it. Then p^k divides a and b, and hence u and v, and so divides $gcd(u, v)$. Hence $gcd(a, b)|gcd(u, v)$.

Suppose that $gcd(u, v) = \prod p^r$. Then p^{r+1} divides neither a nor b and p^r divides at least one of a and b, say a. Then, as p^r divides $u = a + b$ and a, it follows that p^r divides b, and therefore divides $gcd(a, b)$. Hence $gcd(u, v)|gcd(a, b)$. The result follows.

452. (a) Let m be a positive integer. Show that there exists a positive integer k for which the set

$$\{k + 1, k + 2, \ldots, 2k\}$$

contains exactly m numbers whose binary representation has exactly three digits equal to 1.

(b) Determine all intgers m for which there is exactly one such integer k.

Solution 1. (a) For each positive integer k, let $f(k)$ be the number of integers in the set

$$\{k + 1, k + 2, \ldots, 2k\}$$

whose binary representation has exactly three digits equal to 1. When we move from $k - 1$ to k, the set corresponding to $k - 1$ drops the number k and adds the numbers $2k - 1$ and $2k$ to for the set correponding to k. Since k and $2k$ have exactly the same number of ones in their binary representations, we find that, for $k \geq 2$,

$$f(k) = f(k - 1)$$

when $2k-1$ does not have three digits equal to one, and

$$f(k) = f(k-1) + 1$$

when $2k-1$ has exactly three digits equal to one (*i.e.*, has the form $2^a + 2^b + 2^c$ for distinct nonnegative integers a, b, c. There are infinitely many numbers of this form.

Hence $f(k)$ increases by 0 or 1 with every unit increase in k and takes arbitrarily large value. Since $f(1) = 0$, the function f assumes every nonnegative integer.

(b) Suppose that $f(k)$ assumes some value m exactly once. Then, there must be a positive integer r for which $f(r-1) = m-1$, $f(r) = m$ and $f(r+1) = m+1$, so that $2r-1 = 2^t + 2^s + 1$ for some positive integers t and s with $t > s > 0$ (so $t \geq 2$) and the binary representation of $2r+1 = 2^t + 2^s + 2 + 1$ has exactly three digits equal to 1. This can happen only of $s = 1$, so that $2r-1 = 2^t + 3$, $2r+1 = 2^t + 5$ and $r = 2^{t-1} + 2$.

We count the number of integers with three unit binary digits in

$$\{2^{t-1} + 2 + 1, 2^{t-1} + 2^2, \cdots, 2^t, 2^t + 1, 2^t + 2, 2^t + 2 + 1, 2^t + 2^2\} \ .$$

This set includes all the numbers with exactly t digits, except for 2^{t-1} and $2^{t-1} + 1$, neither of which has three unit digits, and exactly $\binom{t-1}{2}$ of them have three unit digits (corresponding to all possible choices of pairs of digit positions). There is one additional number $2^t + 2 + 1$ with three digits. Hence $f(k)$ assumes the value m exactly once if and only if m has the form $1 + \binom{n}{2}$ and $k = 2^n + 2$.

Solution 2. (a) [A. Remorov] Let $k = 2^a + 2^b + 1$, where $a > b \geq 1$. There are $\binom{a}{2}$ numbers with exactly three unit binary digits between 2^a and 2^{a+1} inclusive, since there are a positions in which to place the last two unit digits. There are $\binom{b}{2}$ numbers between 2^a and $2^a + 2^b$ inclusive, since there are b positions available for the last two unit digits. Thus there are $\binom{a}{2} - \binom{b}{2}$ numbers with three unit digits between $2^a + 2^b$ and $2^{a+1} - 1$ inclusive, and so

$$\binom{a}{2} - \binom{b}{2} - 1$$

numbers with three unit digits between $k+1 = 2^a + 2^b + 2$ and $2^{a+1} - 1$ inclusive (the number $k = 2^a + 2^b + 1$ is not included).

There are $\binom{b+1}{2} + 2$ numbers with three unit digits between 2^{a+1} and $2k = 2^{a+1} + 2^{b+1} + 2$ inclusive, since the last two ones can be chosen arbitrarily from the last $b+1$ digits and since $2k-1$ and $2k$ are also included. Hence the number of digits between $k+1$ and $2k$ inclusive is equal to $\binom{a}{2} + b + 1$. Since b can be any integer for which $1 \leq b \leq a-1$, the set of numbers m for which there are exactly m numbers with exactly three unit digits between $k+1$ and $2k$ inclusive contain all the numbers between $\binom{a}{2} + 2$ and $\binom{a}{2} + a = \binom{a+1}{2}$ for $a \geq 2$ (*i.e.*, $3, 5, 6, 8, 9, 10, \cdots$).

There is one such integer when $k = 4$ and two such integers when $k = 6$. When $a \geq 2$ and $k = 2^a + 3$, there are $\binom{a}{2} - 1$ such integers between $2^a + 4$ and $2^{a+1} - 1$ inclusive and also 2 more, $2^{a+1} + 3$ and $2^{a+1} + 6$ for a total of $\binom{a}{2} + 1$ between $k+1$ and $2k$ inclusive. Hence, all values of m can be assumed.

Solution 3. [D. Shi] Let x_m be the mth binary number that contains exactly two digits equal to 1 (so that $x_1 = 3$, $x_2 = 5$, $x_3 = 6$, $x_4 = 9$). We prove that $\{x_m + 1, x_m + 2, \cdots, 2x_m\}$ contains exactly $m-1$ numbers with exactly three unit binary digits.

First, note that there are exactly $n-1$ binary numbers with n digits with exactly two unit digits (the left digit and one other). Suppose that $1 + 2 + \cdots + (n-1) < m \leq 1 + 2 + \cdots + n$, so that $m = \binom{n}{2} + r$ for $1 \leq r \leq n$. Then x_m has $n+1$ binary digits and so $x^m = 2^n + 2^{r-1}$. In the set $\{x_m + 1, \cdots, 2x_m\}$, there are $(r-1) + r + \cdots + (n-1) = \binom{n}{2} - \binom{r-1}{2}$ numbers of the form $2^n + 2^a + 2^b$ with $a \geq r-1, a > b \geq 0$ and $\binom{r}{2}$ numbers of the form $2^{n+1} + 2^a + 2^b$ with $r-1 \geq a > b \geq 0$. Hence there are

$$\binom{n}{r} - \binom{r-1}{2} + \binom{r}{1} = \binom{n}{r} - (r-1) = m-1$$

numbers in $\{x_{m+1}, \cdots, 2x_m\}$ with three unit digits.

The number m of numbers being an increasing function of k, the number $m-1$ is unique if and only if $x_{m+1} = x_m + 1$. This occurs if r is chosen so that $x_m = 2^n + 2^{r-1} + 1$ has two digits equal to 1, which is equivalent to $r = 1$. Hence, the numbers m which occur exactly once are of the form $\binom{n}{2} + 1$ for $n \geq 2$.

453. Let A, B be two points on a circle, and let AP and BQ be two rays of equal length that are tangent to the circle that are directed counterclockwise from their tangency points. Prove that the line AB intersects the segment PQ at its midpoint.

Solution 1. [D. Dziabenko, Y. Wang] If A and B are at opposite ends of a diameter, then AP and BQ are mutual images with respect to a reflection in the centre of the circle and AB bisects PQ at the centre of the circle. Otherwise, wolog, we may suppose that the arc from A to B is less than a semicircle.

Let the lines AP and BQ meet at C and suppose that PA is produced to D so that $DP = 2AP$. Since (in triangle CDQ), $DA : AC = AP : AC = BQ : CB$, $AB \| DQ$. Suppose that AB meets PQ at K. Then (in triangle PDQ), $AK \| DQ$, so that $PA : AD = PK : KQ$. Since $PA = AD$, $PK = KQ$ as desired.

Solution 2. [K. Huynh] The rotation with centre O, the centre of circle, that takes A to B also takes P to Q. Let $\beta = \angle AOP$. Consider the spiral similarity of a rotation about O with angle β followed by a dilation of factor $|OP|/|OA|$. This takes triangle OAB to triangle OPQ and takes the midpoint M of AB to the midpoint N of PQ. Our task is to show that A, B and N are collinear.

Since $OP : OA = ON : OM$ and $\angle AOP = \angle MON = \beta$, triangles OAP and OMN are similar. Hence $\angle OMN = \angle OAP = 90°$. Since triangle OAB is isosceles, $OM \perp AB$, so that $\angle OMB = 90° = \angle OMN$. Hence A, M, B, N are collinear and the lines AB meets the segment PQ at its midpoint.

Solution 3. [P. Chu] Suppose that AB and PQ intersect at M, and that OP and AM intersect at X. We have that $\Delta OAP \sim \Delta OBQ$ and $\Delta OAB \sim \Delta OPQ$. Since $\angle OAB = \angle OPQ$ and $\angle OXA = \angle MXP$, triangles OAX and MPX are similar, and so $AX : OX = PX : MX$. Since, also, $\angle AXP = \angle OXM$, triangles AXP and OXM are similar. Now,

$$\angle MOP + \angle MPO = \angle MOX + \angle QPO = \angle XAP + \angle BAO = 90°$$

whence $\angle OMP = 90°$. Since $OP = OQ$, triangle POQ is isosceles and its altitude OM bisects the base PQ. The result follows.

Solution 4. Let N be the midpoint of PQ. The half-turn (180° rotation) about N interchanges P and Q and takes A to A', so that N is the midpoint of AA'. We show that B lies on AA'.

Let O be the centre of the circle and let $\angle AOB = 2\alpha$. The rotation with centre O that takes A to B also takes P to Q, so that the angle between AP and BQ is equal to 2α. Since AP is carried to $A'Q$ by the half-turn about N, the angle formed by BQ and QA' at Q is equal to 2α. This is an exterior angle to the triangle BQA'.

Since $BQ = PA = PA'$, triangle BQA' is isosceles and so $\angle BA'Q = \angle QBA'$. Hence

$$\angle NAP = \angle A'AP = \angle AA'Q = \angle BA'Q = \frac{1}{2}(\angle BA'Q + \angle QBA') = \alpha \ .$$

However, $\angle BAP$ is equal to the angle between chord and tangent and so equal to half the angle subtended by the chord at the centre O. Hence $\angle BAP = \alpha = \angle NAP$, so that A, B, N are collinear and the result follows.

Solution 5. [C. Sun] Let AB intersect PQ at M. Note that triangle OAB and OPQ are similar isosceles triangles.

$$\begin{aligned}\angle MBO &= 180° - \angle ABO = 180° - (90° - \frac{1}{2}\angle AOB) \\ &= 180° - (90° - \frac{1}{2}\angle POQ) = 180° - \angle PQO \\ &= 180° - \angle MQO \ .\end{aligned}$$

Hence $\angle MBO + \angle MQO = 180°$, so that the quadrilateral $OBMQ$ is concyclic. Therefore $\angle OMQ = \angle OBQ = 90°$, from which $OM \perp PQ$. Because triangle OPQ is isosceles, M is the midpoint of PQ, as desired.

454. Let ABC be a non-isosceles triangle with circumcentre O, incentre I and orthocentre H. Prove that the angle OIH exceeds $90°$.

Solution 1. Suppose that $\angle A > 90°$. Then O and H are both external to the triangle on opposite sides of BC. The points O and H are opposite vertices of a rectangle, two of whose sides are the altitude from A to BC and the right bisector of BC. Since the angle bisector of angle BAC lies between these sides within triangle ABC [why?], I lies inside the rectangle and within the circle of diameter OH. Hence $\angle OIH > 90°$. If $\angle A = 90°$, then O is the midpoint of BC and $H = A$. he same argument can be used (noting that I is not on OH since the triangle is not isosceles).

Suppose that ABC is an acute triangle with $AB < AC < BC$. Let the altitudes be AP, BQ, CR and the medians AL, BM, CN. We have that $AR < AN$, $BP < BL$, $AQ < AM$. Hence H lies inside the quadrilateral $AMON$. Since $\angle RHP > 90°$, $\angle PHC < 90°$. The parallelogram with sides AP, OL, CR, ON has an acute angle at H and O and so is contained in the circle with diameter HO.

Since $AB < AC$, $\angle BAP < \angle CAP$ and $\angle BAL > \angle CAL$, so that the bisector AI of the angle A lies between AP and AL. Similarly, CI lies between CR and CN. Thus I lies within the parallelogram with sides AP, OL, CR, ON and so is contained within the circle of diameter OH. Hence $\angle OIH > 90°$.

Solution 2. Recall some preliminary facts. The nine-point circle of a triangle ABC passes through the midpoints of the sides, the midpoints of the segments joining its vertices to the orthocentre H and the pedal points (*i.e.*, the feet of its altitudes to the sides). Its centre is the midpoint N of the segment joining the the circumcentre O and the orthocentre H of the triangle. Its radius $\frac{1}{2}R$ is equal to half the circumradius R of the triangle ABC and it touches internally the incircle with radius r (as well as all three excircles). (See the book, H.S.M. Coxeter & S.L. Greitzer, *Geometry revisited*, MAA, 1967, §1.8, 5.6). The square of the length of the segment OI is $|OI|^2 = R^2 - 2Rr = R(R - 2r)$ (*ibid*, §2.1)

[Y. Wang] Produce OI to M so that $OI = IM$, and let R and r be be the circumradius and inradius, respectively. Consider triangle OHM. Since N is the midpoint of OH and I is the midpoint of OM, $NI \| HM$ so that $|HM| = 2|NI| = R - 2r$. Since $|IM| = |OI| = \sqrt{R(R-2r)}$ and $\sqrt{R(R-2r)} > R - 2r$, $|IM| > |HM$, so that $\angle IHM > \angle MIH$. Hence $\angle MIH < 90°$ so that $\angle OIH > 90°$.

Solution 3. [D. Dziabenko] See background information in Solution 2. The centre N of the nine-point circle is the midpoint of OH, so that $\overrightarrow{IH} = 2\overrightarrow{IN} - \overrightarrow{IO}$. Since

$$\overrightarrow{IN} \cdot \overrightarrow{IO} = |\overrightarrow{IN}||\overrightarrow{IO}| \cos \angle OIN = \frac{1}{2}(R - 2r)\sqrt{R^2 - 2Rr} \cos \angle OIN \ ,$$

it follows that

$$\begin{aligned} |\overrightarrow{IH}||\overrightarrow{IO}| \cos \angle OIH = \overrightarrow{IH} \cdot \overrightarrow{IO} &= (2\overrightarrow{IN} - \overrightarrow{IO} \cdot \overrightarrow{IO}) \\ &= 2(\overrightarrow{IN} \cdot \overrightarrow{IO}) - |IO|^2 \\ &= (R - 2r)(\sqrt{R^2 - 2Rr}) \cos \angle OIN - (R^2 - 2Rr) \\ &\leq (R - 2r)\sqrt{R^2 - 2Rr} - (R - 2r)R = (R - 2r)[\sqrt{R^2 - 2Rr} - R] < 0 \ . \end{aligned}$$

Hence $\cos \angle OIH < 0$ and so $\angle OIH > 90°$.

455. Let $ABCDE$ be a pentagon for which the position of the base AB and the lengths of the five sides are fixed. Find the locus of the point D for all such pentagons for which the angles at C and E are equal.

Solution 1. [C. Bao] We use analytic geometry, with the assignment $A \sim (0,0)$, $B \sim (1,0)$, $C \sim (a,b)$, $D \sim (x,y)$ and $E \sim (c,d)$. The lengths of the sides are $|AB| = 1$, $|BC| = u$, $|CD| = v$, $|DE| = w$ and $|EA| = t$. We have that $u^2 = (a-1)^2 + b^2$, $v^2 = (x-a)^2 + (y-b)^2$, $w^2 = (x-c)^2 + (y-d)^2$ and $t^2 = c^2 + d^2$.

Now

$$\begin{aligned}\overrightarrow{CB}\cdot\overrightarrow{CD} &= (a-1,b)\cdot(a-x,b-y) = (a-1)(a-x)+b(b-y)\\ &= a^2+b^2-ax-by+x-a\\ &= \frac{1}{2}[(a-1)^2+b^2+(x-a)^2+(b-y)^2-(x-1)^2-y^2]\\ &= \frac{1}{2}[u^2+v^2-(x-1)^2-y^2]\ ,\end{aligned}$$

so that

$$\cos C = \frac{u^2+v^2-[(x-1)^2+y^2]}{2uv}\ .$$

Similarly,

$$\cos E = \frac{w^2+t^2-(x^2+y^2)}{2wt}\ .$$

Hence

$$(u^2+v^2)wt-[(x-1)^2+y^2]wt = (w^2+t^2)uv-[x^2+y^2]uv$$

so that

$$(uv-wt)[x^2+y^2]+2wtx+[(u^2+v^2-1)wt-(w^2+t^2)uv]=0\ .$$

Thus, the point $C \sim (x,y)$ lies on a circle when $uv-wt\neq 0$ and on a straight line perpendicular to AB when $uv=wt$.

456. Let $n+1$ cups, labelled in order with the numbers $0,1,2,\cdots,n$, be given. Suppose that $n+1$ tokens, one bearing each of the numbers $0,1,2,\cdots,n$ are distributed randomly into the cups, so that each cup contains exactly one token.

We perform a sequence of moves. At each move, determine the smallest number k for which the cup with label k has a token with label m not equal to k. Necessarily, $k<m$. Remove this token; move all the tokens in cups labelled $k+1,k+2,\cdots,m$ to the respective cups labelled $k,k+1,m-1$; drop the token with label m into the cup with label m. Repeat.

Prove that the process terminates with each token in its own cup (token k in cup k for each k) in not more that 2^n-1 moves. Determine when it takes exactly 2^n-1 moves.

Solution. Let $(x_0,x_1,x_2,\cdots,x_n)$ denote the arrangement of tokens in which token number x_i is placed in cup i. When $n=0$, token 0 is in cup 0, and $0=2^0-1$ moves are required. When $n=1$, there are two possible distributions of tokens, and at most $1=2^1-1$ moves is needed, with this number required in the case of $(1,0)$. We will establish the result by an induction argument.

First, observe that, for any arrangement $(x_0,x_1,\cdots,x_i,\cdots,x_n)$, any token either remains stationary or moves one cup to the left at each move until it reaches the leftmost cup to the right of tokens already in their cups. Also, note that the number of moves required to first take token x_i to the position from which it first moves to its own cup depends only on the tokens $x_0,\cdots,x_{i-1}$ to the left of it. This can be seen by induction on i. This is clear for $i=1$, since either x_0 will move and x_1 goes to cup 0, or $x_0=0$ and x_1 will move to its own cup. Suppose that this is true for $i=j-1\geq 1$. Then, if $(x_0,x_1,\cdots,x_{j-1})$ is a permutation of $0,1,\cdots,j-1)$, then x_j will remain in position until its left neighbours are sorted, and then will move. Otherwise, x_j will move one position to the left on the first occasion when on of the tokens on the left is moved to the right of it. Since this token is now in cup $j-1$, we can apply the induction hypothesis.

Back to the given problem, we suppose as an induction hypothesis that, for $n=k$, at most 2^k-1 moves are required, and this this number of moves is necessary if and only if the initial arrangement is $(1,2,3,\cdots,k,0)$.

Consider an initial arrangement $(x_0,x_1,\cdots,x_k,x_{k+1})$ in the case $n=k+1$. If $x_{k+1}=k+1$, then this token will never be moved and by the induction hypothesis, the remaining tokens will be put into their proper cups in at most $2^k - 1 < 2^{k+1} - 1$ moves. Suppose that $x_i = k+1$ for $0 \le i \le k$. Consider two initial arrangements:

$$A = (x_0, x_1, \cdots, x_{i-1}, x_i = k+1, x_{i+1}, \cdots, x_{k+1})$$

and

$$B = (x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}) ,$$

where B has $k+1$ tokens numbered from 0 to k inclusive sorted into $k+1$ cups. The number of moves required to move x_i in arrangement A to a position from which it moves to its own cup is equal to the number of moves to move x_{i+1} in arrangement B to a similar position, namely, no more than $2^k - 1$. This number of moves is actually equal to $2^k - 1$ if and only if $B = (1, 2, \cdots, k, 0)$ and $A = (1, 2, \cdots, k, k+1, 0)$ (*i.e.*, $i = k$).

Thus, after at most $2^k - 1$ moves, we have an arrangement with token $k+1$ in cup 0. One additional move takes this token to cup $k+1$ and the rest all in the left cups. Finally, at most $2^k - 1$ moves are required to restore the remaining tokens to their proper cups. Thus, we make at most $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$ moves. This maximum is attained only if we begin with $(1, 2, \cdots, k, k+1, 0)$. The first $2^k - 1$ moves take us to $(k+1, 1, 2, \cdots, k, 0)$; the next move yields $(1, 2, \cdots, k, 0, k+1)$ and the final $2^k - 1$ moves takes us to $(0, 1, 2, \cdots, k, k+1)$.

457. Suppose that $u_1 > u_2 > u_3 > \cdots$ and that there are infinitely many indices n for which $u_n \ge 1/n$. Prove that there exists a positive integer N for which

$$u_1 + u_2 + u_3 + \cdots + u_N > 2006 .$$

Solution. Since there are infinitely many values of n for which $u_n \ge 1/n$, we can select positive integers n_i such that $n_{i+1} > 2n_i$ for $i = 1, 2, 3, \cdots$. Then

$$\sum_{n=n_i+1}^{n_{i+1}} u_n \ge \sum_{n=n_i+1}^{n_{i+1}} u_{n_{i+1}} \ge \frac{n_{i+1} - n_i}{n_{i+1}} > \frac{1}{2}$$

for $i \ge 1$. Let $N = n_{4013}$. Then

$$\sum_{n=1}^{N} u_n \ge \sum_{n=n_1+1}^{n_{4013}} u_n > (4012)(1/2) = 2006 .$$