Solutions for November

465. For what positive real numbers a is

$$\sqrt[3]{2+\sqrt{a}}+\sqrt[3]{2-\sqrt{a}}$$

an integer?

Solution 1. Let $x=\sqrt[3]{2+\sqrt{a}}$, $y=\sqrt[3]{2-\sqrt{a}}$ and $z=x+y$. Then

$$z^3=(x+y)^3=x^3+y^3+3(4-a)^{1/3}z=4+3(4-a)^{1/3}z\ .$$

Hence $27(4-a)z^3=(z^3-4)^3$, whence

$$a=4-\frac{(z^3-4)^3}{27z^3}=\frac{108z^3-(z^3-4)^3}{27z^3}\ .$$

Since $a\geq 0$, z must be either (1) a positive integer for which $108z^3\geq(z^3-4)^3$, or (2) a negative integer for which $108z^3\leq(z^3-4)^3$.

Condition (1) forces $108\geq(z^2-(4/z))^3\geq(z^2-4)^3$, so that $z=1,2$. Condition (2) forces $108\geq(z^2-(4/z))^3\geq z^6$, which is satisfied by no negative integer value of z. Hence, we must have that $(z,a)=(1,5),(2,100/27)$. Since $z=x+y$ is equivalent to $z^3=4+3(4-a)^{1/3}z$, it is straightforward to check that both these answers are correct. Hence $a=5$ or $a=100/27$.

Solution 2. [Yifan Wang] With x and y defined as in the first solution, note that $x>y$ and that $x^3+y^3=(x+y)(x^2-xy+y^2)$. Since $x^2+y^2>(x+y)^2/2$ and $-xy>-(x+y)^2/4$, we have that $4>(x+y)^3/4$, whence $x+y\leq 2$. Since $x^3>-y^3$, $x>-y$, so that $x+y>0$. Hence $x+y=1$ or $x+y=2$.

When $x+y=1$, $x^2-xy+y^2=4$ and so $xy=-1$, and $x=\frac{1}{2}(1+\sqrt{5})$, $y=\frac{1}{2}(1-\sqrt{5})$. Therefore $4-a=x^3y^3=-1$ so that $a=5$.

When $x+y=2$, then $x^2-xy+y^2=2$, so that $xy=2/3$. Therefore $x=\frac{1}{3}(3+\sqrt{3})$, $y=\frac{1}{3}(3-\sqrt{3})$ and $4-a=8/27$. Thus, $a=100/27$. These solutions check out.

Solution 3. [A. Tavakoli] Denote the left side of the equation by $f(a)$. When $a\geq 4$,

$$0\leq f(a)=(\sqrt{a}+2)^{1/3}-(\sqrt{a}-2)^{1/3}=\frac{4}{(\sqrt{a}+2)^{2/3}+(a-4)^{1/3}+(\sqrt{a}-2)^{2/3}}\leq 4^{1/3}<3\ .$$

Let $0\leq a\leq 4$; again $f(a)>0$. Observe that

$$\left(\frac{1}{2}(u+v)\right)^{\frac{1}{3}}\geq\frac{1}{2}u^{\frac{1}{3}}+\frac{1}{2}v^{\frac{1}{3}}$$

for all nonnegative values of u and v. (This can be seen by using the concavity of the function $t^{1/3}$, or from the power-mean inequality $(1/2)(s+t)\leq[(1/2)(s^3+t^3)]^{1/3}$.) Setting $u=\sqrt[3]{2+\sqrt{a}}$ and $v=\sqrt[3]{2-\sqrt{a}}$, we find that $3>2\times 2^{1/3}\geq f(a)>0$ with equality if and only if $a=0$. Hence the only possible integer values of $f(a)$ are 0 and 1.

Let $x=\sqrt[3]{2-\sqrt{a}}$, so that $2+\sqrt{a}=4-x^3$. Then

$$\begin{aligned}f(a)=1&\iff x+(4-x^3)^{1/3}=1\\&\iff 4-x^3=1-3x+3x^2-x^3\\&\iff x^2-x-1=0\iff x=(1\pm\sqrt{5})/2\\&\iff x^3=2\pm\sqrt{5}\ .\end{aligned}$$

The larger root of the quadratic leads to $x^3 > 2$ and so is extraneous. Hence $x^3 = 2 - \sqrt{5}$, and so $\sqrt{a} = \sqrt{5}$, $a = 5$.

$$\begin{aligned} f(a) = 2 &\Longleftrightarrow x + (4 - x^3)^{1/3} = 2 \\ &\Longleftrightarrow 4 - x^3 = (2 - x)^3 = 8 - 12x + 6x^2 - x^3 \\ &\Longleftrightarrow 3x^2 - 6x + 2 = 0 \Longleftrightarrow x = \frac{3 \pm \sqrt{3}}{3} \ . \ . \end{aligned}$$

Now,

$$\left(\frac{3 \pm \sqrt{3}}{3}\right)^3 = 2 \pm \frac{10\sqrt{3}}{9} \ .$$

The larger value of x leads to $x^3 > 2$, and so is inadmissible. The smaller value of x leads to $x^3 = 2 - (10\sqrt{3}/9)$ and $\sqrt{a} = (10\sqrt{3}/9)$, $a = 100/27$. Both values of a check out.

466. For a positive integer m, let $\overline{m}$ denote the sum of the digits of m. Find all pairs of positive integers (m, n) with $m < n$ for which $(\overline{m})^2 = n$ and $(\overline{n})^2 = m$.

Solution. Let $m = m_k \cdots m_1 m_0$ where $0 \le m_i \le 9$ are the digits of m. Then

$$10^k \le m < n = (m_k + \cdots + m_0)^2 \le [(k+1)10]^2 \ ,$$

whence $10^{k-2} \le (k+1)^2$ and $0 \le k \le 3$.

Hence $m < n = (m_3 + m_2 + m_1 + m_0)^2 \le (4 \times 9)^2 = 36^2$. Since m and n are both perfect squares, we need only consider $m = r^2$, where $1 \le r \le 36$.

In the case that $k = 3$, $\overline{m} < 1 + 9 + 9 + 9 = 28$. Since $28^2 < 1000 < m < n$, there are no examples. In the case that $k = 2$, $\overline{m} < 6 + 9 + 9 = 24$ and so $n^2 \le 24^2$. The only possibility is $(m, n) = (169, 256)$. There are no possibilities when $k = 0$ or $k = 1$.

Hence, the only number pair is $(m, n) = (169, 256)$.

Comment. This is problem **621** from *The College Mathematics Journal*.

467. For which positive integers n does there exist a set of n distinct positive integers such that

(a) each member of the set divides the sum of all members of the set, *and*

(b) none of its proper subsets with two or more elements satisfies the condition in (a)?

Solution. When $n = 1$, condition (b) is satisfied vacuously, and any singleton will do. When $n = 2$, such a set cannot be found. If a and b are any two positive integers, then condition (b) entails that both a and b divide $a + b$, and so must divide each other. This cannot happen when a and b are distinct.

When $n \ge 3$, a set of the required type can be found. For example, let

$$S_n = \{1, 2, 2 \times 3, 2 \times 3^2, \cdots 2 \times 3^{n-3}, 3^{n-2} \ .$$

The sum of the elements in S_n is $2 \times 3^{n-2}$, which is divisible by each member of S_n.

Consider any proper subset R of S_n with at least three numbers. If 3^{n-2} belongs to R, then the sum of the elements of R must be strictly between 3^{n-2} and $2 \times 3^{n-2}$, and so not divisible by 3^{n-2}. If R does not contain 3^{n-2}, then its largest entry has the form 2×3^k with $1 \le k \le n - 3$. Then the sum of R is greater than 2×3^k and does not exceed $1 + 2(1 + 3 + \cdots + 3^k) = 3^{k+1} < 2(2 \times 3^k)$. Hence this sum is not divisible by 2×3^k. As we have seen, no doubleton satisfies the condition. Hence (b) is satisfied for all subsets of S_n.

Comment. This is problem **1504** in the October, 1996 issue of *Mathematics Magazine*.

468. Let a and b be positive real numbers satisfying $a + b \geq (a - b)^2$. Prove that

$$x^a(1-x)^b + x^b(1-x)^a \leq \frac{1}{2^{a+b-1}}$$

for $0 \leq x \leq 1$, with equality if and only if $x = \frac{1}{2}$.

Comment. Denote the left side by $f(x)$. When $a = b$, $f(x) = 2x^a(1-x)^a$, which is maximized when $x = 1/2$, its maximum value being 2×4^{-a}. In the general case, the solution can be obtained by calculus. Since $f(0) = f(1) = 0$ and the function possesses a derivative everywhere, the maximum occurs when $f'(x) = 0$ and $0 < x < 1$. Wolog, assume that $a < b$. We have that

$$f'(x) = x^{a-1}(1-x)^{a-1}[(a-(a+b)x)(1-x)^{b-a} + (b-(a+b)x)x^{b-a}] \ .$$

This solution can be found in *Mathematics Magazine* 70:4 (October, 1997), 301-302 (Problem **1505**), and is fairly technical. It would be nice to have a more transparent argument. Is there a solution that avoids calculus, at least for rational a and b?

A second solution, employs the substitution $2x = 1 - y$ to get the equivalent inequality

$$(1-y)^a(1+y)^b + (1-y)^b(1+y)^a \leq 2$$

for $|y| \leq 1$. Wolog, we can let $a = b + c$ with $c \geq 0$. Then the condition becomes $2b \geq c^2 - c$. Then the inequality is equivalent to

$$(1-y^2)^b[(1-y)^c + (1+y)^c] \leq 2 \ ,$$

for $|y| \leq 1$.

Let $0 \leq c \leq 1$. Then, for $t > 0$, the function t^c is concave, so that, for $u, v > 0$,

$$\left(\frac{u+v}{2}\right)^c \geq \frac{u^c + v^c}{2} \ .$$

Setting $(u, v) = (1 - y, 1 + y)$, we find that $(1-y)^c + (1+y)^c \leq 2$ for $|y| \leq 1$. Hence the inequality holds, with equality occurring when $y = 0$ $(x = 1/2)$.

When $c > 1$, I do not have a clean solution. First, it suffices to consider the inequality when b is replaced by $\frac{1}{2}(c^2 - c)$. Thus, we need to establish that

$$(1-y^2)^{(1/2)(c^2-c)}[(1-y)^c + (1-y)^c] \leq 2 \qquad (*)$$

for $|y| \leq 1$. The derivative of the natural logarithm of the left side is a positive multiple of

$$g(y) = (1+y)^c(1-cy) - (1-y)^c(1+cy) \ .$$

If this can be shown to be nonpositive, then the result will follow. An equivalent inequality is

$$\left(1 - \frac{2y}{1+y}\right)^2 = \left(\frac{1-y}{1+y}\right)^c \geq \left(\frac{1-cy}{1+cy}\right) = \left(1 - \frac{2cy}{1+cy}\right) \ ,$$

for $c > 1$ and $|y| \leq 1$.

469. Solve for t in terms of a, b in the equation

$$\sqrt{\frac{t^3 + a^3}{t + a}} + \sqrt{\frac{t^3 + b^3}{t + b}} = \sqrt{\frac{a^3 - b^3}{a - b}}$$

where $0 < a < b$.

Solution 1. The equation is equivalent to

$$\sqrt{t^2 - at + a^2} + \sqrt{t^2 - bt + b^2} = \sqrt{a^2 + ab + b^2} \ .$$

Square both sides of the equation, collect the nonradical terms on one side and the radical on the other and square again. Once the polynomials are expanded and like terms collected, we obtain the equation

$$0 = t^2(a+b)^2 - 2ab(a+b)t + a^2b^2 = [t(a+b) - ab]^2 \ ,$$

whence $t = ab/(a+b)$. This can be checked by substituting it into the equation.

Solution 2. [Y. Wang] As in solution 1, we can find an equivalent equation, which can then be manipulated to

$$\sqrt{(t-(a/2))^2 + (\sqrt{3}a/2)^2} + \sqrt{(t-(b/2))^2 + (-\sqrt{3}b/2)^2} = \sqrt{(a/2 - b/2)^2 + (\sqrt{3}a/2 + \sqrt{3}b/2)} \ .$$

If we consider the points $A \sim (a/2, \sqrt{3}a/2)$, $B \sim (b/2, -\sqrt{3}b/2)$ and $T \sim (t, 0)$, then we can interpret this equation as stating that $AT + BT = AB$. By the triangle inequality, we see that T must lie on AB, so that the slopes of AT and BT are equal. Thus

$$\frac{\sqrt{3}a}{a - 2t} = \frac{\sqrt{3}b}{2t - b} \ ,$$

whence $t = ab/(a+b)$.

470. Let ABC, ACP and BCQ be nonoverlapping triangles in the plane with angles CAP and CBQ right. Let M be the foot of the perpendicular from C to AB. Prove that lines AQ, BP and CM are concurrent if and only if $\angle BCQ = \angle ACP$.

Solution 1. [A. Tavakoli] Let BP and AQ intersect at K. Let $\angle BCQ = \alpha$, $\angle ACP = \beta$ and $\angle BCA = \gamma$. By the trigonometric form of Ceva's theorem, CM, AP and BQ are concurrent if and only if

$$\frac{\sin \angle BCM}{\sin \angle ACM} \cdot \frac{\sin \angle KAC}{\sin \angle KAB} \cdot \frac{\sin \angle KBA}{\sin \angle KBC} = 1 \ . \qquad (1)$$

This holds whether K lies inside or outside of the triangle.

We have that $\sin \angle BCM = \cos \angle CBA$, $\sin \angle ACM = \cos \angle CAB$, and, by the Law of Sines applied to triangles ACQ and ABQ,

$$\sin \angle KAC = \sin \angle QAC = (\sin \angle ACQ)(|QC|)/(|AQ|) \ ,$$

and

$$\sin \angle KAB = \sin \angle QAB = (\sin \angle ABQ)(|QB|)/(|AQ|) \ .$$

Therefore

$$\frac{\sin \angle KAC}{\sin \angle KAB} = \left(\frac{\sin \angle ACQ}{\sin \angle ABQ}\right) \cdot \left(\frac{|QC|}{|QB|}\right) = \left(\frac{\sin(\gamma + \alpha)}{\sin(\angle ABC + 90^\circ)}\right) \cdot \left(\frac{1}{\sin \alpha}\right) = \frac{-\sin(\gamma + \alpha)}{(\cos \angle CBA) \sin \alpha} \ .$$

Similarly,

$$\sin \angle KBA = \sin \angle BAP(|AP|/|BP|)$$

$$\sin \angle KBC = \sin \angle BCP(|PC|/|BP|)$$

and so

$$\frac{\sin \angle KBA}{\sin \angle KBC} = \frac{\sin(\angle BAC + 90^\circ)}{\sin(\beta+\gamma)} \cdot \frac{|AP|}{|PC|} = \frac{-\cos(\angle BAC)\sin\beta}{\sin(\beta+\gamma)} \ .$$

Hence the condition for concurrency becomes

$$\frac{\sin(\gamma+\alpha)}{\sin\alpha} \cdot \frac{\sin\beta}{\sin(\gamma+\beta)} = 1$$

$$\Longleftrightarrow \sin\gamma\cot\alpha + \cos\gamma = \sin\gamma\cot\beta + \cos\gamma$$

$$\Longleftrightarrow \cot\alpha = \cot\beta \Longleftrightarrow \angle BCQ = \alpha = \beta = \angle ACP \ .$$

This is the required result.

Solution 2. We do some preliminary work. Suppose that PB and AQ intersect at O, and that X and Y are the respective feet of the perpendiculars from C to PB and AQ. Since $\angle CXP = \angle CAP = 90^\circ$, $CAXP$ is concyclic and so $\angle ACP = \angle AXP$. Similarly $CQBY$ is concyclic and so $\angle BCQ = \angle BYQ$. Since $\angle CXO = \angle CYO = 90^\circ$, X and Y lie on the circle with diameter CO. Hence $\angle YCO = \angle YXO = \angle YXB$.

Now suppose that $\angle BCQ = \angle ACP$. Let CO produced meet AB at N. Since $\angle AXP = \angle ACP = \angle BCQ = \angle BYQ$, it follows that $\angle AXB = \angle AYB$ so that $BYXA$ is concyclic and so $\angle YXB = \angle YAB$. Therefore

$$\angle YCN = \angle YCO = \angle YXB = \angle YAB = \angle YAN$$

and $ANYC$ is concyclic/ Hence $\angle CNA = \angle CYA = 90^\circ$ and N must coincide with M.

On the other hand, let CM pass through O. Since $\angle CYA = \angle CMA = 90^\circ$, $AMYC$ is concyclic so that

$$\angle YAB = \angle YAM = \angle YCM = \angle YCO = \angle YXB \ .$$

Therefore $BAXY$ is concyclic and $\angle BXA = \angle BYA \Rightarrow \angle AXP = \angle BYQ$. Since $CAXP$ and $CYBQ$ are concyclic, $\angle ACP = \angle AXP = \angle BYQ = \angle BCQ$.

471. Let I and O denote the incentre and the circumcentre, respectively, of triangle ABC. Assume that triangle ABC is not equilateral. Prove that $\angle AIO \le 90^\circ$ if and only if $2BC \le AB + CA$, with equality holding only simultaneously.

Solution 1. Wolog, let $AB \ge AC$. Suppose that the circumcircle of triangle ABC intersects AI in D. Construct the circle Γ with centre D that passes through B and C. By the symmetry of AB and AC in the angle bisector AD, this circle intersects segment AB in a point F such that $AF = AC$. Let Γ intersect AD at P. Then chords CP and FP have the same length. If $AB > AC$, this implies that P is on the angle bisector of angle ABC. If $AB = AC$, then $\angle ABC = \angle ADC = \angle PDC = 2\angle PBC$. In either case, $P = I$.

Let E be on the ray BA produced such that $AE = AC$. Since $\angle DAC = \frac{1}{2}\angle BAC = \angle AEC$ and $\angle ADC = \angle ABC = \angle EBC$, triangles ADC and EBC are similar, and so

$$ID : AD = CD : AD = BC : BE = BC : (AB + AC) \ .$$

But $\angle AIO \le 90^\circ$ if and only if $ID/AD \le 1/2$, and so is equivalent to $2BC \le AB + AC$, with equality holding only simultaneously. (Solution due to Wu Wei Chao in China.)

Solution 2. We have that $\angle AIO \le 90^\circ$ if and only if $\cos\angle AIO \ge 0$, if and only if $|AO|^2 \le |OI|^2 + |IA|^2$. Let a, b, c be the respective sidelengths of BC, CA, AB; let R be the circumradius and let r be the inradius of triangle ABC. Since, by Euler's formula, $|OI|^2 = R^2 - 2Rr$, and $r = |IA|\sin(A/2)$, the foregoing inequality is equivalent to

$$2R \le \frac{r}{\sin^2(A/2)} = \frac{2r}{1-\cos A} \ .$$

Applying $R = a/(2\sin A)$, $r = bc\sin A/(a+b+c)$ and $2bc\cos A = b^2+c^2-a^2$, we find that

$$\begin{aligned} r - R(1-\cos A) &= \frac{bc\sin A}{a+b+c} - \frac{a(1-\cos A)}{2\sin A} \\ &= \sin A\left[\frac{bc}{a+b+c} - \frac{a(1-\cos A)}{2\sin^2 A}\right] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[2bc + 2bc\cos A - a(a+b+c)] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[2bc + b^2 + c^2 - a^2 - a(a+b+c)] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[(b+c)^2 - 2a^2 - a(b+c)] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[(b+c+a)(b+c-2a)] \ . \end{aligned}$$

Hence the inequality $R(1-\cos A) \leq r$ is equivalent to $2a \leq b+c$. The desired result follows. (Solution due to Can A. Minh, USA)

Solution 3. [Y. Wang] Let AI intersect the circumcircle of triangle ABC at D. Since AI bisects the angle BAC and the arc BC, we have that $BD = BC$. Also,

$$\angle DIC = \angle CAD + \angle ACI = \angle BCD + \angle BCI = \angle DCI \ ,$$

whence $DC = DI = DB$. Using Ptolemy's Theorem, we have that

$$AB \times CD + BD \times AC = AD \times BC \ ,$$

so that

$$AB \times DI + DI \times AC = (AI + ID) \times BC \ .$$

Hence

$$k \equiv \frac{AB + AC}{BC} = 1 + \frac{AI}{ID} \ .$$

If $AB = AC$, then A, O, I are collinear. Let $k < 2$; then $AI < ID$ and I lies between A and O and $\angle AIO = 180°$. Let $k > 2$; then $AI > ID$, O lies between A and I and $\angle AIO = 0°$. [If $k = 2$, then $AI = ID$, the incentre and circumcentre coincide and the triangle is equilateral – the excluded case.]

Wolog, suppose that $AB > AC$. Then the circumcentre O lies within the triangle ABD. Let P be the foot of the perpendicular from O to AD. Then P is the midpoint of AD and the angle AIO is greater than, equal to or less than $90°$ according as I is in the segment AP, coincides with P or is in the segment PD. These correspond to $k < 2$, $k = 2$ and $k > 2$, and the result follows.