Solutions for June

444. (a) Suppose that a 6×6 square grid of unit squares (chessboard) is tiled by 1×2 rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

(b) Is the same thing true for an 8×8 array?

(c) Is the same thing true for a 6×8 array?

Solution. (a) There are 18 dominoes and 10 interior lines in the grid. For the decomposition not to occur, each of the lines must be straddled by at least one domino. We argue that, in fact, at least two dominos must straddle each line. Since no domino can straddle more than one line, this would require 20 dominos and so yield a contradiction.

Each interior line has six segments. For a line next to the side of the square grid, an adjacent domino between it and the side must either cross one segment or be adjacent to two segments. Since the number of segments is even, evenly many dominos must cross a segment. For the next line in, an adjacent domino must be adjacent to two segments, be adjacent to one segment and cross the previous line, or cross one segment. Since the number of dominoes straddling the previous line is even, there must be evenly many that cross the segment. In this way, we can work our way from one line to the next.

Comment. F. Barekat had the following argument. Consider a subrectangle determined by one interior line. It contains an even number of unit squares. Since each domino covers two unit squares, there must be an even number of unit squares belonging to dominoes that straddle the internal line.

(b) Number the squares in the grid by pairs ij of digits where the square is in the ith row and jth column. Here is a tiling with dominos in which each interior line is straddled and no decomposition into subrectangles is possible:

$$(11-12),(13-14),(15-16),(17-27),(18-28),(21-31),(22-32),(23-33),$$

$$(24-34),(25-26),(35-45),(36-46),(37-38),(41-42),(43-44),(47-57),$$

$$(48-58),(51-61),(52-53),(54-55),(56-66),(62-72),(63-73),(64-74),$$

$$(65-75),(67-68),(71-81),(76-77),(78-88),(82-83),(84-85),(86-87)\ .$$

(c) [J. Schneider; C. Sun] Using a similar notation as in (b), we have the example for which no decomposition into subrectangles is possible. Note that there are 12 interior lines and 24 dominoes, so that, for each example, each interior line is straddled by exactly two dominoes.

$$(11-21),(12-13),(14-15),(16-26),(17-18),(22-32),(23-24),(25,35),$$

$$(27-28),(31-41),(33-34),(36-37),(38-48),(42-43),(44-54),(45-46),$$

$$(47-57),(51-52),(53-63),(55-56),(58-68),(61-62),(64-65),(66-67)\ .$$

Here are two coverings that exhibit symmetry due respectively to P. Chen and K. Huynh. The first is

$$(11-21),(12-22),(13-14),(15-16),(17-18),(23-33),(24-25),(26-27),$$

$$(28-38),(31-32),(34-44),(35-45),(36-37),(41-51),(42-43),(46-56),$$

$$(47-48),(52-53),(54-55),(57-67),(58-68),(61-62),(63-64),(65-66)\ .$$

The second is

$$(11-21),(12-13),(14-15),(16-17),(18-28),(22-32),(23-24),(25-26),$$

$$(27-37),(31-41),(33-34),(35-36),(38-48),(42-43),(44-54),(45-55),$$
$$(46-47),(51-52),(53-63),(56-66),(57-58),(61-62),(64-65),(67-78)\ .$$

445. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

Solution 1. Wolog, we may assume that the parabolas have the equations $y = ax^2$ and $y = b(x-1)^2 + c$. The common chord has equation

$$a[b(x-1)^2 + c - y] - b[ax^2 - y] = 0\ ,$$

or

$$(a-b)y + 2abx - a(b+c) = 0\ . \tag{1}$$

Consider a point (u, au^2) on the first parabola. The tangent at this point has equation $y = 2aux - au^2$. The abscissa of the intersection point of this tangent with the parabola of equation $y = b(x-1)^2 + c$ is given by the question

$$bx^2 - 2(b+au)x + (au^2 + b + c) = 0\ .$$

This has coincident roots if and only if

$$(b+au)^2 = b(au^2 + b + c) \Longleftrightarrow a(a-b)u^2 + 2abu - bc = 0\ . \tag{2}$$

In this situation, the coincident roots are $x = 1 + (au)/b$ and the point of contact of the common tangent with the second parabola is

$$\left(1 + \frac{au}{b}, \frac{a^2u^2}{b} + c\right)\ .$$

The midpoint of the segment joining the two contact points is

$$\left(\frac{b + au + bu}{2b}, \frac{abu^2 + a^2u^2 + bc}{2b}\right)\ .$$

Plugging this into the left side of (1) and using (2) yields that

$$\begin{aligned}[1/(2b)][(a-b)a(a+b)u^2 + (a-b)bc + 2ab^2 + 2ab(a+b)u - 2ab(b+c)]\\ = [(a+b)/(2b)][a(a-b)u^2 + 2abu - bc] = 0\ .\end{aligned}$$

Thus, the coordinates of the midpoint of the segment satisfy (1) and the result follows.

Solution 2. [A. Feizmohammadi] Let the two parabolas have equations $y = ax(x-u)$ and $y = bx(x-v)$. Since the two parabolas must open the same way for the situation to occur, wolog, we may suppose that $a, b > 0$. The parabolas intersect at the points $(0,0)$ and $((au-bv)/(a-b), (ab(au-bv)(u-v)/(a-b)^2)$, and the common chord has equation $(a-b)y - ab(u-v)x = 0$.

Let $y = mx+k$ be the equation of the common tangent. Then both of the equations $ax^2-(au+m)x-k = 0$ and $bx^2 - (bv+m)x - k = 0$ have double roots. Therefore $(au+m)^2 + 4ak = (bv+m)^2 + 4bk = 0$, from which (by eliminating k),

$$ab(au^2 - bv^2) + 2ab(u-v)m + (b-a)m^2 = 0\ .$$

The common tangent of equation $y = mx + k$ touches the first parabola at

$$\left(\frac{au+m}{2a}, \frac{m^2 - a^2u^2}{4a}\right)$$

and the second parabola at

$$\left(\frac{bv+m}{2b}, \frac{m^2-b^2v^2}{4b}\right) .$$

The midpoint of the segment joining these two points is

$$\left(\frac{ab(u+v)+(a+b)m}{4ab}, \frac{(a+b)m^2-ab(au^2+bv^2)}{8ab}\right) .$$

Using these coordinates as the values of x and y, we find that

$$\begin{aligned}
8ab[(a-b)y-ab(u-v)x] &= (a-b)[(a+b)m^2-ab(au^2+bv^2)]-2a^2b^2(u-v)(u+v) \\
&\qquad -2ab(a+b)(u-v)m \\
&= (a^2-b^2)m^2-2ab(a+b)(u-v)m \\
&\qquad -[(a-b)a^2bu^2+(a-b)ab^2v^2+2a^2b^2u^2-2a^2b^2v^2] \\
&= (a^2-b^2)m^2-2ab(a+b)(u-v)m-[a^3bu^2+a^2b^2u^2-a^2b^2v^2-ab^3v^2] \\
&= (a+b)[(a-b)m^2-2ab(u-v)m]-[ab(a(a+b)u^2-ab(b(a+b)v^2] \\
&= (a+b)[(a-b)m^2-2ab(u-v)m-ab(au^2-bv^2)]=0 .
\end{aligned}$$

Solution 3. [J. Kileel] We may assume that both parabolas have vertical axes and that one has equation $y=x^2$. The second has an equation of the form $y=ax^2+bx+c$, where $a>0$ and $a\neq 1$. (The latter ensures two points of intersection.)

The equation of the chord through the points (d,d^2) and (e,e^2) is $y=(d+e)x-de$. The abscissae x_1 and x_2 of the intersection points of the two parabolas are the roots of the quadratic $(a-1)x^2+bx+c=0$, so that

$$x_1+x_2=\frac{-b}{a-1} \qquad \text{and} \qquad x_1x_2=\frac{c}{a-1} .$$

The line passing through the points (x_1,x_1^2) and (x_2,x_2^2) is

$$(1-a)y=bx+c . \tag{1}$$

The equation of a line tangent to the first parabola at (u,u^2) is

$$y=2ux-u^2 , \tag{2}$$

and to the second parabola at (v,av^2+bv+c) is

$$y=(2av+b)x+(c-av^2) . \tag{3}$$

For the common tangent, these two equations are identical, whence

$$2u=2av+b \tag{4}$$

and

$$u^2=av^2-c . \tag{5}$$

Eliminating v from (4) and (5) yields

$$4(1-a)u^2-4bu-4ac+b^2=0 . \tag{6}$$

The chord and common tangent intersect at the point

$$\left(\frac{au^2 - u^2 - c}{2au - 2u + b}, -\frac{2uc + bu^2}{2au - 2u + b}\right) .$$

(Solve (1) and (2).) The midpoint of the segment of the common tangent joining the two points of tangency has, by (4), the abscissa,

$$\frac{u+v}{2} = \frac{2au + 2u - b}{4a} .$$

Now

$$\begin{aligned}\frac{au^2 - u^2 - c}{2au - 2u + b} &- \frac{2au + 2u - b}{4a} \\ &= \frac{4a^2u^2 - 4au^2 - 4ac - 4a^2u^2 + (2u - b)^2}{4a(2au - 2u + b} \\ &= \frac{4(1-a)u^2 - 4bu - 4ac + b^2}{4a(2au - 2u + b)} = 0 ,\end{aligned}$$

by (6). Hence the abscissae of the intersection point and the midpoint of the common tangent are equal and the result follows.

Comment. This can be solved using projective geometry, as it holds for any conic. To capture the idea of midpoint, we use harmonic range involving points at infinity.

446. Suppose that you have a 3×3 grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

Two players A and B play a game. They take alternate turns, A putting a 0 in any unoccupied square of the grid and B putting a 1. The first player is A, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then B can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

(a) What is the maximum number of legitimate moves possible in a game?

(b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

(c) Which player has a winning strategy? Explain.

Solution. (a) A game cannot continue to nine moves. Otherwise, the line sum must be three times the value on the centre square of the grid (why?) and so must be 0 or 3. But some line must contain both zeros and ones, yielding a contradiction. [An alternative argument is that, if the array is filled, not all the rows can have the same numbers of 0s and 1s, and therefore cannot have the same sums.] However, an 8-move grid is possible, in which one player selects the corner squares and the other the squares in the middle of the edges.

(b) After three or fewer moves have occurred, there are at most three additional squares that would complete a line and the next player can avoid all of these. Consider any game after four moves have occurred and it is A's turn to play a zero. Suppose, first of all, that no lines have been filled with numbers. The only way an inaccessible square can occur is if it is the intersection of two lines each having the other two squares filled in. This can happen in at most one way. So A would have at least four possible squares to fill in. On the other hand, if three of the first four moves complete a line, the fourth number can bar at most three squares for A in the three lines determined by the fourth number and one of the other three. Thus, A would have at least two possible positions to fill. Thus, a game must go to at least five moves.

A five-move game can be obtained when A has placed three 0's down the left column and B has 1 in the centre square and another square of the middle column. Each remaining position is closed to B as it would complete a line whose sum is not 0. Other configurations where no further move is possible are the one in which the top row starts with two 1s and the middle row has three 0s, and where the top row has three 0s and the bottom row has a 1 at each end.

(c) A has a winning strategy. A places 0 in the middle square. For the next four moves, he plays symmetrically, completing the line through the centre initiated by B. After the fifth move, up to rotation and reflection, there are four possible configurations (where * denotes a vacant square, and the rows are listed from left to right):

$$\alpha : (1, 1, *\,/\,*, 0, *\,/\,*, 0, 0)$$

$$\beta : (1, *, 1\,/\,*, 0, *\,/\,0, *, 0)$$

$$\gamma : (1, *, *\,/\,0, 0, 1\,/\,*, *, 0)$$

$$\delta : (*, 1, *\,/\,0, 0, 1\,/\,*, 0, *)$$

In the case of α, A can respond symmetrically to B if B plays in the middle row, and can achieve

$$(1, 1, *\,/\,*, 0, 0\,/\,1, 0, 0)$$

if B plays in the bottom row. B cannot move. In the case of β, B has only one move and A can counter by a move in the middle row. As for γ and δ, A can respond respond symetrically to B and then B has no further move.

447. A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax-b} - \sqrt{cx-d} = 1\ .$$

Solution 1. Solving the general equation properly leads to

$$\sqrt{ax-b} - \sqrt{cx-d} = 1 \Longrightarrow ax - b = 1 + cx - d - 2\sqrt{cx-d}$$

$$\Longrightarrow (a-c)x = (b+1-d) - 2\sqrt{cx-d}\ .$$

To make the manipulation simpler, specialize to $a = c + 1$ and $d = b + 1$. Then the equation becomes

$$x^2 = 4(cx - d) \Longrightarrow 0 = x^2 - 4cx + 4d\ .$$

Using the student's "method" to solve the same equation gives $ax - b - cx - d = 1$ which yields $x = (1 + b + d)/(a - c) = 2d$. So, for the "method" to work, we need

$$0 = 4d^2 - 8cd + 4d = 4d(d - 2c + 1)$$

which can be achieved by making $2c = d + 1$. So we can take

$$(a, b, c, d) = (t + 1, 2(t - 1), t, 2t - 1)$$

for some real t. The original problem corresponds to $t = 2$.

The equation

$$\sqrt{(t+1)x - 2(t-1)} - \sqrt{tx - (2t-1)} = 1$$

is satisfied by $x = 2$ and $x = 4t - 2$. The first solution works for all values of t, while the second is valid if and only if $t \geq \frac{1}{2}$. The equation $(t + 1)x - 2(t - 1) - tx - (2t - 1) = 1$ is equivalent to $x = 4t - 2$.

Solution 2. [G. Goldstein] *Analysis.* We want to solve simultaneously the equations

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \tag{1}$$

and

$$ax - b - cx - d = 1\ . \tag{2}$$

¿From (1), we find that

$$ax - b = 1 + (cx - d) + 2\sqrt{cx - d}\ . \tag{3}$$

¿From (2) and (3), we obtain that $d = \sqrt{cx - d}$, so that $x = (d^2 + d)/c$. From (2), we have that $x = (1 + b + d)/(a - c)$.

Select a, c, d so that $d > 0$ and $ac(a - c) \neq 0$, and choose b to satisfy

$$\frac{d^2 + d}{c} = \frac{1 + b + d}{a - c}\ .$$

Let

$$x = \frac{d^2 + d}{c} = \frac{1 + b + d}{a - c} = \frac{(d^2 + d) + (1 + b + d)}{c + (a - c)} = \frac{(d + 1)^2 + b}{a}\ .$$

Then

$$\sqrt{ax - b} - \sqrt{cx - d} = \sqrt{(d + 1)^2} - \sqrt{d^2} = (d + 1) - d = 1$$

and

$$ax - b - cx - d = (d + 1)^2 + b - b - d^2 - d - d = 1\ .$$

Comments. In Solution 2, if we take $c = d = 1$, we get the family of parameters $(a, b, c, d) = (a, 2a - 4, 1, 1)$. R. Barrington Leigh found the set of parameters given in Solution 1. A. Feizmohammadi and Y. Wang provided the parameters $(a, b, c, d) = (2c, 0, c, 1)$. J. Schneider took $(a, b, c, d) = (n, n - 4, 2, 1)$, with both equations satisfied by $x = 1$. A. Tavakoli had the paramatrization $(a, b, c, d) = (2c, b, c, \sqrt{1 + b}) = (2c, d^2 - 1, c, d)$, with $c \neq 0$ and $b > -1$. A. Remorov offered $(a, b, c, d) = (a, 1, 2a/5, 1)$ with $a \neq 0$ and $x = 5/a$, while D. Shi offered $(a, b, c, d) = (d + 2, d + 1, d, d)$ with $d > 0$. But C. Sun had the simplest family of all with $(a, b, c, d) = (a, b, 0, 0)$ with $a > 0$ and $b > 0$.

If $a = c$, then $ax - b - cx - d = 1$ is satisfied by any value of x as long as $1 + b + d = 0$. The surd equation becomes

$$\sqrt{ax - b} - \sqrt{ax + (b + 1)} = 1$$

$$\Longrightarrow ax - b = ax + (b + 1) + 1 + 2\sqrt{ax + (b + 1)}$$

$$\Longrightarrow b + 1 = -\sqrt{ax + (b + 1)}$$

$$\Longrightarrow (b + 1)^2 = ax + (b + 1) \Longrightarrow x = \frac{(b + 1)b}{a}\ .$$

Since $ax - b = b^2$ and $ax + (b+1) = (b+1)^2$, we should take $b \leq -1$ in order to satisfy the equation. This is a singular case in which the linear equation has infinitely many solutions and the surd equation either zero or one solution.

448. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

Solution 1. It will take 20 hours to catch the criminal, so that he is at large for 30 hours. Let t be the time in hours from the time the pursuit begins until the time of capture, and let x be the speed of the criminal in miles per hour. Then

$$(10 + t)x = t(x + 3) \Longrightarrow 10x = 3t \ .$$

Consider the situation 2.4 hours before the pursuers met the express. The distance between the pursuers and criminal is the distance the pursuers can travel in 4.8 hours, namely $4.8(x + 3)$ miles (note that the train and pursuers travel equal distances during the 2.4 to a common meeting point). Since the relative speed of the pursuers relative to the criminal is 3 miles per hour, it will take the pursuers an additional $(1/3)(4.8)(x + 3) = 1.6(x + 3)$ hours to close in. Hence

$$t = (8 - 2.4) + 1.6(x + 3) \Longrightarrow t = 10.4 + 1.6x \ .$$

Since $x = 0.3t$, it follows that $104 = 0.52t$ and $t = 20$.

Solution 2. We note that the information that the pursuers travel 3 miles per hour faster than the criminal turns out to be redundant. Let u be the speed of the criminal and v be the speed of the pursuers and of the freight train. The distance from the prison to the place where the freight train encountered the criminal is

$$8v + (2.4)v = (18 - 2.4)u = 15.6u \Longrightarrow 10.4v = 15.6u \Longrightarrow 2v = 3u \ .$$

After 8 hours of pursuit, the distance between the criminal and his pursuers is

$$18u - 8v = t(v - u)$$

for some value of t. This equation reduces to

$$(18 + t)u = (t + 8)v$$

which with the earlier equation yields $t = 12$. Thus after 8 hours, the distance between the criminal and his pursuers is $18u - 8v = 12(v - u)$. Since the pursuers are travelling at speed $v - u$ relative to the criminal, it will take them 12 hours to close the gap. Hence it will take 20 hours after the pursuers begin to catch the criminal.

Solution 3. [J. Schneider] Let t be the elapsed time in hours since the escape of the criminal, u the speed of the criminal in miles per hour, and z the distance from the prison at time t in miles. For convenience, project the train's motion so that it reaches the prison at time s; since the train takes the same length of time (8 hours) to reach the prison from its encounter with the pursuers as the pursuers take to reach the train, $s = 10 + 8 + 8 = 26$. (This can also be obtained from equating $(u + 3)8 = -(u + 3)(18 - s)$.)

When $t \geq 10$, the criminal is distant from the prison vt miles, the pursuers $(u + 3)(t - 10)$ miles and the train $-(u + 3)(t - 26)$ miles. Since the train and criminal meet at $t = 15.6$,

$$15.6u = -(u + 3)(15.6 - 26) = 10.4(u + 3) \ ,$$

so that $3u = 2(u+3)$ and $u = 6$. When the pursuers catch the criminal, we have that $6t = 9(t-10)$, so that $t = 30$. Thus, it takes the pursuers 20 hours to catch the criminal.

Solution 4. [F. Ban] Let the speed of the criminal be u miles per hour and of the pursuers $u+3$ miles per hour. When the criminal met the express, both were $15.6u$ miles from the prisoner and the pursuers were $5.6(u+3)$ miles from the prison. Since the express and the pursuers went at the same speed, they met at a distance $8(u+3)$ from the prison, which is exactly halfway between the former position of the pursuers at $5.6(u+3)$ and of the criminal at $15.6u$. Thus

$$15.6u - 8(u+3) = 8(u+3) - 5.6(u+3) \Longrightarrow 15.6u = 10.4(u+3) \Longrightarrow u = 6, u+3 = 9 \ .$$

If the time taken to catch the criminal after the pursuers start out is t, then the distance from the prison at time t is

$$9t = (10+t)6 \Longrightarrow t = 20 \ ,$$

so that it takes the pursuers 20 hours after setting out to catch the criminal.

Comment. P. Chen set up the equation

$$8(u+3) + 2.4(u+3) = 15.6u \ .$$

449. Let $S = \{x : x > -1\}$. Determine all functions from S to S which both

(a) satisfies the equation $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$, and

(b) $f(x)/x$ is strictly increasing or strictly decreasing on each of the two intervals $\{x : -1 < x < 0\}$ and $\{x : x > 0\}$.

Solution. We first check that the function is viably defined. Suppose such a function $f(x)$ exists. Then for all $x, y \in S$,

$$1 + x + f(y) + xf(y) = (1+x)(1+f(y)) > 0 \ ,$$

so that $x + f(y) + xf(y) \in S$. Similarly, $y + f(x) + yf(x) \in S$.

If we set $x = y$, then we have, for all $x \in S$.

$$f(x + f(x) + xf(x)) = x + f(x) + xf(x) \ .$$

Thus, there is at least one number $a \in S$ for which $f(a) = a$. Let $b = a + f(a) + af(a) = 2a + a^2 = a(a+2) = (a+1)^2 - 1$. Then $b \in S$ and $f(b) = b$.

Suppose, if possible, that $a > 0$; then $b > a > 0$. However, $f(a)/a = f(b)/b = 1$, which contradicts condition (b). Suppose, if possible, that $-1 < a < 0$; then $-1 < b < a < 0$. However, as before, we get a contradiction with condition (b). Hence, the only remaining possibility is that $a = 0$, so that $x + f(x) + xf(x) = 0$ for all $x \in S$. Hence

$$f(x) = \frac{-x}{1+x} = \frac{1}{1+x} - 1$$

for $x > -1$.

We verify that this function works. We have that

$$x + f(y) + xf(y) = (1+x)(1+f(y)) - 1 = \frac{1+x}{1+y} - 1$$

and

$$y + f(x) + yf(x) = \frac{1+y}{1+x} - 1 \ .$$

Hence

$$f(x + f(y) + xf(y)) = \frac{(1 + y) - (1 + x)}{1 + x} = y + f(x) + yf(x) \ .$$

Note that $f(x)/x = -1/(1 + x)$ increases on $S\backslash\{0\}$.

Comment. We can establish more directly that $f(0) = 0$. First, note that $f(x)$ is one-one. Suppose that $f(a) = f(b)$. Then, for all $x \in S$, $x + f(a) + xf(a) = x + f(b) + xf(b)$. Aplying f yields that $a + f(x) + af(x) = b + f(x) + bf(x)$, so that $(a - b)(1 + f(x)) = 0$. Hence $a = b$. For all $x \in S$, $f(x + f(0) + xf(0)) = f(x)$, so that $f(0)(1 + x) = 0$ and $f(0) = 0$. One immediate consequence of this is that $f(f(x)) = x$ for all $x \in S$.

K. Huynh began by letting $f(0) = k$. Setting $(x, y) = (0, 0)$ led to $f(k) = k$, and $(x, y) = (0, k)$ to $0 = k(k + 1)$, whence $k = 0$. Thus $f(0) = 0$, from which $f(f(y)) = y$ for all $y \in S$.

450. The 4-sectors of an angle are the three lines through its vertex that partition the angle into four equal parts; adjacent 4-sectors of two angles that share a side consist of the 4-sector through each vertex that is closest to the other vertex.

Prove that adjacent 4-sectors of the angles of a parallelogram meet in the vertices of a square if and only if the parallelogram has four equal sides.

Solution 1. Let the parallelogram be $ABCD$ and let its diagonals intersect at P. Suppose that $ABCD$ has equal sides, so that $ABCD$ is a rhombus and its diagonals right bisect each other. The intersection of the adjacent 4-sectors are the respective incentres I, J, K, L of the triangle ABP, BCP, CDP, DAP. Since IP, JP, KP, LP bisect the respective angles APB, BPC, CPD, DPA, it follows that each of the angles IPJ, JPK, KPL, LPI is equal to $90°$. The four triangles APB, BPC, CPD, DPA are congruent with corresponding angles at P. Hence $IP = JP = KP = LP$ and $IJKL$ is square.

Suppose that $IJKL$ is square. Observe that $\angle BAI + \angle ABI$, being one quarter of the sum of adjacent angles of a parallelogram, is equal to $45°$, whence $\angle AIB = 135°$. Similarly, $\angle BJC = 135°$. Wolog, assume that $AB \geq AD = BC$. Then, by similar triangles, BAI and BCI, we have that $BJ \leq BI$, so that $\angle BIJ \leq \angle BJI$. Similarly, $AI \geq AL$, so that $\angle CJK = \angle ALI \leq \angle AIL$. Hence

$$\begin{aligned} 90° = \angle LIJ &= 360° - (\angle AIL + \angle AIB + \angle BIJ) \\ &\geq 360° - (\angle CJK + \angle BJC + \angle BJI) = \angle IJK = 90° \ . \end{aligned}$$

We must have equality throughout, so that

$$\angle BIJ = \angle BJI \Longrightarrow BI = BJ \Longrightarrow AB = BC$$

and $ABCD$ is a rhombus.

Solution 2. [A. Tavakoli] Use the same notation as in Solution 1. Suppose that the angles at A and C are 4α and at B and D are 4β. Since the angle sum of the parallelogram is $360°$, $\alpha + \beta = 45°$. Hence

$$\angle AIB = \angle BJC = \angle CKD = \angle DLA = 135° \ .$$

¿From the Sine Law, we have that

$$|BI| = \sqrt{2}(\sin\alpha)a \qquad |BJ| = \sqrt{2}(\sin\alpha)b$$

$$|AI| = \sqrt{2}(\sin\beta)a \qquad |AL| = \sqrt{2}(\sin\beta)b \ .$$

Assume that $IJKL$ is a square. Then, since $|IJ| = |IL|$, the Cosine Law reveals that

$$|BI|^2 + |BJ|^2 - 2|BI||BJ|\cos 2\beta = |AI|^2 + |AL|^2 - 2|AI||AL|\cos 2\alpha \ ,$$

whence

$$2a^2\sin^2\alpha + 2b^2\sin^2\alpha - 4ab\sin^2\alpha\cos 2\beta = 2a^2\sin^2\beta + 2b^2\sin^2\beta - 4ab\sin^2\beta\cos 2\alpha$$

$$\Longrightarrow (a^2+b^2)(\sin^2\alpha - \sin^2\beta) = 2ab[\sin^2\alpha(1-2\sin^2\beta) - \sin^2\beta(1-2\sin^2\alpha)] = 2ab(\sin^2\alpha - \sin^2\beta)\ .$$

If $\alpha = \beta$, the parallelogram is a rectangle. Suppose that $AD \geq AB$ and P is the centre of the rectangle. Compairing similar triangles AIB and ALD, we see that $AL \geq AI$, with equality if and only if $AD = AB$. Hence, since $\angle IAL = 45°$,

$$\angle AIL \geq 67\frac{1}{2}^\circ \geq \angle ALI \Rightarrow \angle LIP \geq 45° \leq \angle LIP$$

with equality if and only if $AD = AB$. Hence, $IJKL$ is a square if and only if $ABCD$ is a square.

Otherwise, $(a-b)^2 = a^2 + b^2 - 2ab = 0$ and $a = b$.

Assume that $ABCD$ is a rhombus, so that $b = a$. The triangles ABI, CBJ, ADL are congruent (ASA), so that $AI = AL$ and $BI = BJ$. Observe that, by the angle sum of triangles, $2\alpha + 2\angle AIL = 2\beta + 2\angle BIJ = 180°$. Hence

$$\angle LIJ = 360° - (\angle AIL + \angle AIB + \angle BIJ) = 360° - (90° - \alpha + 135° + 90° - \beta) = 90°\ .$$

By the Sine Law applied to triangle AIL, we have that

$$|IL| = \sin 2\alpha \cdot \frac{|AI|}{\sin(90° - \alpha)} = \frac{2\sin\alpha\cos\alpha\sqrt{2}(\sin\beta)a}{\cos\alpha} = 2\sin\alpha\sin\beta\sqrt{2}a\ .$$

Similarly, $|IJ| = |JK| = |KL| = 2\sin\alpha\sin\beta\sqrt{2}a = |IL|$. Hence $IJKL$ is a square.

Solution 3. [K. Huynh (first part)] Suppose that $ABCD$ is a rhombus. Then its diagonals bisect its opposite angles and $ABCD$ has reflective symmetry about each diagonal. The four triangles AIB, CJB, CKD, ALD are congruent (ASA); in particular, $BI = BJ = DK = DL$. The reflection about BD takes $A \leftrightarrow C$, $I \leftrightarrow J$ and $K \leftrightarrow L$, so that AC, IJ, LK are perpendicular to BD and $IL = JK$. Similarly BD, LI and JK are perpendicular to AC, and $IJ = KL$. Hence $IJKL$ is a rectangle.

Since BI bisects $\angle ABD$, I us equidistant from AB and BD. Similarly, I is equidistant from AB and AC. Hence, I is equidistant from AC and BD. Since $|IJ|$ is twice the distance from I to BD and $|IL|$ twice the distance from I to AC, $IJ = JK$ and so $IJKL$ is a square.

The reverse implication can be proved by a contradiction argument. Suppose that $AD > AB$. Determine point D' on segment AD and point C' on segment BC so that $ABC'D'$ is a rhombus. Let $IJKL$ be the internal figure determined by the 4-sectors for $ABCD$ and $I'J'K'L'$ for $ABC'D'$. Observe that $I' = I$, that $IJ'K'L'$ is a square and that L and J both lie within the angle $L'I'J'$ (since $D'L' \| DL$ and $C'J' \| CJ$). Hence $\angle LIJ < \angle L'IJ' = 90°$, so that $IJKL$ is not a square. Thus, if $IJKL$ is a square, then $AD = AB$ and the result follows.