Solutions to February problems

430. Let triangle ABC be such that its excircle tangent to the segment AB is also tangent to the circle whose diameter is the segment BC. If the lengths of the sides BC, CA and AB of the triangle form, in this order, an arithmetic sequence, find the measure of the angle ACB.

Solution. Let M be the midpoint of BC, E the centre of the excircle tangent to AB and r its radius, T, S and K the respective points of tangency of this excircle to CB, CA and AB respectively, and P the point of tangency of the two circles of the problem. Following convention, let a, b, c be the sidelengths of the sides of triangle ABC and s its semiperimeter. We have that $|CS| = |CT| = s$, $|AK| = |AS| = s - b$ and $|BK| = |BT| = s - a$. Then

$$[ABC] = [ACE] + [BCE] - [ABE] = \frac{1}{2}rb + \frac{1}{2}ra - \frac{1}{2}rc = r(s - c) \ .$$

Consider the right triangle MET. We have that $|ME| = |MP| + |PE| = \frac{a}{2} + r$, $|MT| = |MB| + |BC| = \frac{a}{2} + s - a = s - \frac{a}{2}$ and $|ET| = r$, so that

$$\left(r + \frac{a}{2}\right)^2 = r^2 + \left(s - \frac{a}{2}\right)^2 \Longrightarrow ra = s(s - a) \ .$$

Using Heron's formula, we find that

$$r(s - c) = [ABC] = \sqrt{s(s - a)(s - b)(s - c)} \ ,$$

so that

$$\frac{s(s - a)(s - c)}{a} = \sqrt{s(s - a)(s - b)(s - c)} \Longrightarrow s(s - a)(s - c) = a^2(s - b) \ .$$

Since a, b, c are in arithmetic progression, for some number d, $a = b - d$, $c = b + d$ and $2s = 3b$, so that

$$\begin{aligned} \frac{3b}{2}\left(\frac{b}{2} - d\right)\left(\frac{b}{2} + d\right) &= (b - d)^2\left(\frac{b}{2}\right) \\ &\Longrightarrow 3b(b^2 - 4d^2) = 4b(b - d)^2 \\ &\Longrightarrow 3b^2 - 12d^2 = 4b^2 - 8bd + 4d^2 \\ &\Longrightarrow 0 = b^2 - 8bd + 16d^2 = (b - 4d)^2 \ . \end{aligned}$$

Hence $(a, b, c) = (3d, 4d, 5d)$ so that triangle ABC is right and $\angle C = 90°$.

431. Prove the following trigonometric identity, for any natural number n:

$$\sin\frac{\pi}{4n + 2} \cdot \sin\frac{3\pi}{4n + 2} \cdot \sin\frac{5\pi}{4n + 2} \cdots \sin\frac{(2n - 1)\pi}{4n + 2} = \frac{1}{2^n} \ .$$

Solution 1. [F. Barekat] Let $\theta = \pi/(4n + 2)$. For each integer m, we have that

$$\sin m\theta = \frac{\sin 2m\theta}{2\cos m\theta} = \frac{\sin 2m\theta}{2\sin(\frac{\pi}{2} - m\theta)} \ .$$

When $m = 2k - 1$ $(1 \le k \le n)$,

$$\frac{\pi}{2} - m\theta = \frac{\pi}{2}\left[1 - \frac{2k - 1}{2n + 1}\right] = \frac{\pi}{2}\left[\frac{2(n + 1 - k)}{2n + 1}\right] = 2(n + 1 - k)\theta \ .$$

Hence

$$\begin{aligned}\prod_{k=1}^{n} \sin \frac{(2k-1)\pi}{4n+2} &= \prod_{k=1}^{n} \sin(2k-1)\theta = \prod_{k=1}^{n} \frac{\sin 2(2k-1)\theta}{2\sin 2(n+1-k)\theta} \\ &= \frac{\prod_{k=1}^{n} \sin 2(2k-1)\theta}{\prod_{k=1}^{n} 2\sin 2(n+1-k)\theta} \\ &= \frac{1}{2^n}\frac{\prod_{k=1}^{n} \sin 2(2k-1)\theta}{\prod_{k=1}^{n} \sin 2k\theta} = \frac{1}{2^n}\prod_{k=1}^{n} \frac{\sin 2(2k-1)\theta}{\sin 2k\theta} \\ &= \frac{1}{2^n}\prod\left\{\frac{\sin 2i\theta}{\sin 2i\theta} : i \;\; \text{odd}, 1 \le i \le n\right\} \cdot \prod \left\{\frac{\sin 2(2n+i-j)\theta}{\sin 2j\theta} : j \;\; \text{even}, 2 \le j \le n\right\} \ .\end{aligned}$$

Since $2(2n+1-j)\theta + 2j\theta = (4n+2)\theta = \pi$, all fractions in the products are equal to unity, and the required value is $1/2^n$.

Solution 2. We first illustrate the argument for $n = 3$. Consider a regular heptagon $ABCDEFG$ with $|AB| = a$, $|AC| = b$ and $|AD| = c$, Let M be the midpoint of DE so that AM right bisects DE as well as the parallel diagonals CF and BG. Being one-quarter of the angle subtended by a side at the centre of the circumcircle, $\angle DAM = \pi/14$. Since $\frac{1}{2}|BC| = |AB| \sin \angle BAM$, $\frac{1}{2}b = a \sin \frac{5\pi}{14}$. Since $\frac{1}{2}|CF| = |AC| \sin \angle CAM$, $\frac{1}{2}c = b \sin \frac{3\pi}{14}$. Since $\frac{1}{2}|DE| = |AD| \sin \angle DAM$, $\frac{1}{2}a = c \sin \frac{\pi}{14}$. Multiplying these equations yields that

$$\frac{1}{2^3} = \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \ .$$

In general, consider a regular $(2n+1)-$gon $AB_1B_2 \cdots B_nC_n \cdots C_2C_1$, with M the midpoint of B_nC_n. Observe that the segments B_kC_k $(1 \le k \le n)$ are all parallel. Suppose that $\theta = \pi/(4n+2)$. Then, for $1 \le k \le n$,

$$\angle B_kAM = \frac{[2n-(2k-1)]\pi}{4n+2} = [2n-(2k-1)]\theta = [2(n+1-k)-1]\theta \ .$$

Therefore,

$$\frac{1}{2}|B_kC_k| = |AB_k| \sin[2(n+1-k)-1]\theta \ .$$

The chord AB_k has $k-1$ vertices of the polygon on one side and $2n-k$ vertices on the other side, while the chord B_kC_k has $2k-1$ vertices of the polygon on one side and $2(n-k)$ vertices on the other. Hence $|AB_k| = |B_jC_j|$ where $j = \frac{1}{2}[2n+1-k]$ when k is odd and $j = \frac{1}{2}k$ when k is even. The sequence $\{|AB_k|\}$ is a permutation of the sequence $\{|B_kC_k|\}$. Hence

$$\begin{aligned}\frac{1}{2^n}\prod_{k=1}^{n} |AB_k| = \frac{1}{2^n}\prod_{k=1}^{n} |B_kC_k| &= \prod_{k=1}^{n} \frac{1}{2}|B_kC_k| \\ &= \prod_{k=1}^{n} |AB_k| \sin[2(n+1-k)-1]\theta = \prod_{k=1}^{n} |AB_k| \prod_{k=1}^{n} \sin(2k-1)\theta \ ,\end{aligned}$$

from which the result follows.

432. Find the exact value of:

(a)

$$\sqrt{\frac{1}{6} + \frac{\sqrt{5}}{18}} - \sqrt{\frac{1}{6} - \frac{\sqrt{5}}{18}} \ ;$$

(b)

$$\sqrt{1+\frac{2}{5}} \cdot \sqrt{1+\frac{2}{6}} \cdot \sqrt{1+\frac{2}{7}} \cdot \sqrt{1+\frac{2}{8}} \cdots \sqrt{1+\frac{2}{57}} \cdot \sqrt{1+\frac{2}{58}} \ .$$

Solution. (a)

$$\sqrt{\frac{1}{6}+\frac{\sqrt{5}}{18}} = \frac{1}{6}\sqrt{6+2\sqrt{5}} = \frac{1}{6}(1+\sqrt{5})$$

and

$$\sqrt{\frac{1}{6}-\frac{\sqrt{5}}{18}} = \frac{1}{6}\sqrt{6-2\sqrt{5}} = \frac{1}{6}(\sqrt{5}-1) \ ,$$

so that the difference is equal to 1/3.

(b) For each n, $\sqrt{1+(2/n)} = \sqrt{(n+2)/n}$, so that the product is equal to

$$\sqrt{\frac{7}{5}\cdot\frac{8}{6}\cdot\frac{9}{7}\cdots\frac{59}{57}\cdot\frac{60}{58}} = \sqrt{\frac{59}{5}\cdot\frac{60}{6}} = \sqrt{118} \ .$$

433. Prove that the equation

$$x^2 + 2y^2 + 98z^2 = 77777\ldots777$$

does not have a solution in integers, where the right side has 2006 digits, all equal to 7.

Solution. Since, modulo 7, squares have the values 0, 1, 2, 4, $x^2+2y^2 \equiv 0 \pmod 7$ implies that $x \equiv y \equiv 0 \pmod 7$, whence $x = 7u$ and $y = 7v$ for some integers u and v. Hence

$$0 \equiv 7u^2 + 14v^2 + 14z^2 = 11111\ldots111$$

(mod 7). However $111111 = 7 \times 15873 \equiv 0 \pmod 7$, and $2006 = 6 \times 334 + 2$. Thus,

$$11111\ldots111 = 11 + 111111(10^2 + 10^8 + \cdots + 10^{2000}) \equiv 11 \equiv 4$$

(mod 7), and we arrive at a contradiction.

434. Find all natural numbers n for which $2^n + n^{2004}$ is equal to a prime number.

Solution. Let $N = 2^n + n^{2004}$. If n is even, then N is even and composite. Let n be odd and not a multiple of 3. Then $N \equiv 2^n + (n^2)^{1003} \equiv 2 + 1 \equiv 0 \pmod 3$. When $n = 1$, $N = 3$ and is prime, while when $n > 1$, N exceeds 3 and is composite.

Finally, let n be a multiple of 3. Then $n = 3k$ for some integer k and $N = (2^k)^3 + (n^{668})^3$ is properly divisible by $2^k + n^{668}$. Hence N is prime exactly when $n = 1$.

435. A circle with centre I is the incircle of the convex quadrilateral $ABCD$. The diagonals AC and BD intersect at the point E. Prove that, if the midpoints of the segments AD, BC and IE are collinear, then $AB = CD$.

Solution. Let M be the midpoint of AD and N the midpoint of BC. Since the mindpoints of AD, BC and IE are collinear, I and E are on opposite sides of MN. Wolog, let I lie inside $AMNB$ and E lie inside $DMNC$.

We have that

$$\begin{aligned}
[MIN] &= [AMNB] - [ABI] - [AMI] - [BNI] \\
&= [AMNB] - [ABI] - \frac{1}{2}([ADI] + [BCI]) \\
&= [AMNB] - \frac{1}{2}[ABI] - \frac{1}{2}([ABI] + [ADI] + [BCI]) \\
&= [AMNB] - \frac{1}{2}[ABI] - \frac{1}{2}([ABCD] - [CDI]) \ .
\end{aligned}$$

Similarly,

$$[MEN] = [DMNC] - \frac{1}{2}[DCE] - \frac{1}{2}([ABCD] - [ABE]) \ .$$

Since MN bisects IE, I and E are equidistant from MN and $[MIN] = [MEN]$. Now

$$[AMNB] = [AMN] + [ABN] = \frac{1}{2}[AND] + \frac{1}{2}[ABC]$$

and

$$[DMNC] = [MDN] + [DNC] = \frac{1}{2}[AND] + \frac{1}{2}[DBC] \ .$$

Hence

$$[AND] + [ABC] - [ABI] - [ABCD] + [CDI] = [AND] + [DBC] - [DEC] - [ABCD] + [ABE] \ ,$$

whence

$$[ABC] + [CDI] + [DEC] = [DBC] + [ABE] + [ABI] \ .$$

Since

$$[ABC] + [DEC] = [ABCDE] = [DBC] + [ABE] \ ,$$

$[CDI] = [ABI]$. But I is equidistant from CD and AB whence $AB = CD$,

436. In the Euro-African volleyball tournament, there were nine more teams participating from Europe than from Africa. In total, the European won nine times as many points as were won by all of the African teams. In this tournamet, each team played exactly once against each other team; there were no ties; the winner of a game gets 1 point, the loser 0. What is the greatest possible score of the best African team?

Solution. Let a be the number of teams from Africa, so that $a+9$ is the number of teams from Europe. Supose that there were k African wins over Europeans. Then the total number of points taken by the African teams is $\binom{a}{2} + k = \frac{1}{2}a(a-1) + k$, while the Europeans won

$$\binom{a+9}{2} + [a(a+9) - k] = \frac{(a+9)(3a+8)}{2} - k \ .$$

By the given conditions,

$$\frac{(a+9)(3a+8)}{2} - k = 9\left[\frac{a(a-1)}{2} + k\right] \ ,$$

which simplifies to $3a^2 - 22a + (10k - 36) = 0$. This is a quadratic equation in a with discriminant equal to $916 - 120k = 4(229 - 30k)$. There are integer values of a satisfying the quadratic only if $229 - 30k$ is square with $k > 0$. Thus, $k = 2$ or $k = 6$.

When $k = 2$, $0 = 3a^2 - 22a - 16 = (a-8)(3a+2)$ so $a = 8$. In this case, there are 8 African teams, and any of these teams can get at most $7 + 2 = 9$ points, which can occur when one African team vanquishes all the other African teams as well as two European teams.

When $k = 6$, $0 = 3a^2 - 22a + 24 = (a-6)(3a-4)$ so $a = 6$. In this case, there are 6 African teams, and any of these teams can get at most $5 + 6 = 11$ points, which can occur when one African team vanquishes all the other African teams as well as six European teams.

Thus the greatest possible score for an African team is 11.