PROBLEMS FOR JUNE

Please send your solution to

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no later than August 15, 2006. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

444. (a) Suppose that a 6×6 square grid of unit squares (chessboard) is tiled by 1×2 rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

(b) Is the same thing true for an 8×8 array?

(c) Is the same thing true for a 6×8 array?

445. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

446. Suppose that you have a 3×3 grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

Two players A and B play a game. They take alternate turns, A putting a 0 in any unoccupied square of the grid and B putting a 1. The first player is A, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then B can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

(a) What is the maximum number of legitimate moves possible in a game?

(b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

(c) Which player has a winning strategy? Explain.

447. A high school student asked to solve the surd equation

$$\sqrt{3x - 2} - \sqrt{2x - 3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation
$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \ .$$

448. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way,

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they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

449. Let $S = \{x : x > -1\}$. Determine all functions from S to S which both

(a) satisfies the equation $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$, and

(b) $f(x)/x$ is strictly increasing or strictly decreasing on each of the two intervals $\{x : -1 < x < 0\}$ and $\{x : x > 0\}$.

450. The 4-sectors of an angle are the three lines through its vertex that partition the angle into four equal parts; adjacent 4-sectors of two angles that share a side consist of the 4-sector through each vertex that is closest to the other vertex.

Prove that adjacent 4-sectors of the angles of a parallelogram meet in the vertices of a square if and only if the parallelogram has four equal sides.