Solutions for October Problems

Comment on problems 339 and 342. In both these problems, a condition was left out and made each of them trivial. Accordingly, problem 339 is marked out of 4 and problem 342 out of 3, for the basic solution. However, additional marks were provided for students who recognized that the problems might have been misstated and provided work that led to the intended solutions. While I try to make sure that the problems are correct, and certainly on contests, the problems are generally gone over very carefully, mistakes do occur. If on a contest, you feel that a mistake has been made in formulating a problem, then you should state clearly a *non-trivial* version of the problem and solve that. In the solutions below, the corrected version of the problem is given.

339. Let a, b, c be integers with $abc \neq 0$, and u, v, w be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0 \ .$$

Let r be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable for rational values of x, y, z.

Solution 1. Suppose, wolog, that $u \neq 0$. Try a solution of the form

$$(x, y, z) = (u(1 + t), vt, wt) \ .$$

Then $au^2 + 2au^2t + au^2t^2 + b^2v^2t^2 + cw^2t^2 = r$ implies that $2au^2t = r - au^2$, from which we find the value $t = (r - au^2)/((2au^2)$. Since a, b, c, u, v, w, r, t are all rational, so is the trial solution.

Solution 2. [R. Peng] Suppose that $r = p/q$. Then the equation with r on the right is satisfied by

$$(x, y, z) = \left(\frac{p}{2} + \frac{1}{2qa}, \left(\frac{p}{2} - \frac{1}{2qa}\right)\left(\frac{v}{u}\right), \left(\frac{p}{2} - \frac{1}{2qa}\right)\left(\frac{w}{u}\right)\right) \ .$$

Comment. If not all a, b, c are zero, then it is trivial to prove that the equation with r on the right has a solution; the only rôle that the equation with 0 on the right plays is to ensure that a, b, c do not all have the same sign. However, some made quite heavy weather of this. The hypothesis that integers are involved should alert you to the fact that some special character of the solution is needed. It is unreasonable to ask that the solution be in integers, but one could seek out rational solutions.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

Solution. The smallest number of combinations that will guarantee success is 32. Denote the eight positions of each whell by the digits 0, 1, 2, 3, 4, 5, 6, 7, so that each combination can be represented by an ordered triple (a, b, c) of three digits. We show that a suitably selected set of 32 combinations will do the job. Let $A = \{(a, b, c) : 0 \leq a, b, c \leq 3 \text{ and } a + b + c \equiv 0 (\text{mod })4\}$ and $B = \{(u, v, w) : 4 \leq u, v, w \leq 7 \text{ and } u + v + w \equiv 0 (\text{mod })4\}$. Any entry in the triples of A and B is uniquely determined by the other two, and any ordered pair is a possibility for these two. Thus, each of A and B contains exactly 16 members. If (p, q, r) is any combination, then either two of p, q, r belong to the set $\{0, 1, 2, 3\}$ and agree with corresponding entries in a combination in A, or two belong to $\{4, 5, 6, 7\}$ and agree with corresponding entries in a combination in B.

We now show that at least 32 combinations are needed. Suppose, if possible, that a set S of combinations has three members whose first entry is 0: $(0, a, b), (0, c, d), (0, e, f)$. There will be twenty-five combinations of the form $(0, y, z)$ with $y \neq a, c, e,\ z \neq b, d, f$, that will not match in two entries any of these three. To cover such combinations, we will need at least 25 distinct combinations of the form (x, y, z) with $1 \leq x \leq 7$. None of the 28 combinations identified so far match the $7 \times 6 = 42$ combinations of the form (u, v, w), where $v \in \{a, c, e\}$, $w \in \{b, d, f\}$, $(v, w) \neq (a, b), (c, d), (e, f)$. Any combination of the form (u, v, t) or (u, t, w) can cover at most three of these and of the form (t, v, w) at most 7. Thus, S will need at least $37 = 3+25+(42/7)$ members to cover all the combinations. A similar argument obtains if there are only three members in S with any other given entry. If there are only one or two members in S with a given entry, say first entry 0, then at least 36 combinations would be needed to cover all the entries with first entry 0 and the other entries differing from the entries of these two elements of S.

Thus, a set of combinations will work only if there are at least four combinations with a specific digit in each entry, in particular at least four whose first entry is k for each of $k = 0, \cdots, 7$. Thus, at least 32 entries are needed.

Comment. Some solvers formulated the problem in terms of the minimum number of rooks (castles) required to occupy or threaten every cell of a solid $8 \times 8 \times 8$ chessboard.

341. Let s, r, R respectively specify the semiperimeter, inradius and circumradius of a triangle ABC.

(a) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in geometric progression.

Comment. In the solutions, we will use the following facts, the establishment of which is left up to the reader:

$$a + b + c = 2s$$
$$ab + bc + ca = s^2 + 4Rr + r^2$$
$$abc = 4Rrs$$

An efficient way to get the second of these is to note that the square of the area is given by $r^2s^2 = s(s-a)(s-b)(s-c)$ from which

$$r^2s = s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc = s^3 - 2s^3 + (ab+bc+ca)s - 4Rrs\ .$$

Solution 1. (a) a, b, c are in arithmetic progression if and only if

$$\begin{aligned}
0 &= (2a-b-c)(2b-c-a)(2c-a-b) \\
&= (2s-3a)(2s-3b)(2s-3c) \\
&= 8s^3 - 12s^2(a+b+c) + 18s(ab+bc+ca) - 27abc \\
&= 2s^3 - 36Rrs + 18r^2s\ .
\end{aligned}$$

Since $s \neq 0$, the necessary and sufficient condition that the three sides be in arithmetic progression is that $s^2 + 9r^2 = 18Rr$.

(b) First, note that

$$\begin{aligned}
a^3 + b^3 + c^3 &= (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc \\
&= 2s^3 - 12Rrs - 6r^2s\ ,
\end{aligned}$$

and

$$\begin{aligned}
a^3b^3 + b^3c^3 + c^3a^3 &= (ab+bc+ca)^3 - 3abc(a+b+c)(ab+bc+ca) + 3(abc)^2 \\
&= (s^2 + 4Rr + r^2)^3 - 24Rrs^4 - 48R^2r^2s^2 - 24Rr^3s^2\ .
\end{aligned}$$

a, b, c are in geometric progression if and only if

$$\begin{aligned} 0 &= (a^2 - bc)(b^2 - ca)(c^2 - ab) \\ &= abc(a^3 + b^3 + c^3) - (a^3b^3 + b^3c^3 + c^3a^3) \\ &= 32Rrs^4 - (s^2 + 4Rr + r^2)^3 \ , \end{aligned}$$

The necessary and sufficient condition is that

$$(s^2 + 4Rr + r^2)^3 = 32Rrs^4 \ .$$

Solution 2. The three sides of the triangle are the three real roots of the cubic equation

$$x^3 - 2sx^2 + (s^2 + r^2 + 4Rr)x - 4Rrs = 0 \ .$$

The three sides are in arithmetic progression if and only if one of them is equal to $2s/r$ and are in geometric progression if and only if one of them is equal to their geometric mean $\sqrt[3]{4Rrs}$.

(a) The condition is that $2s/3$ satisfies the cubic equation:

$$0 = 8s^3 - 6s(4s^2) + 9(s^2 + r^2 + 4Rr)(2s) - 108Rrs = 2s(s^2 + 9r^2 - 18Rr) \ .$$

(b) The condition is that $\sqrt[3]{4Rrs}$ satisfies the cubic equation: $2s(4Rrs)^{1/3} = s^2 + 4Rr + r^2$ or $32Rrs^3 = (s^2 + 4Rr + r^2)^3$.

Solution 3. [B.H. Deng] Assume that b lies between a and c, inclusive. (a) The three sides are in arithmetic progression if and only if $b = \frac{2}{3}s$ or $a + c = 2b$. Since $4Rrs = abc$, this is equivalent to $6Rr = ac$, which in turn is equivalent to

$$r^2 + s^2 + 4Rr = (a + c)b + ac = 2b^2 + ac = (8/9)s^2 + 6Rr$$

or $s^2 + 9r^2 = 18Rr$.

(b) The three sides are in geometric progression if and only if $b^3 = abc = 4Rrs$ and $ac = b^2$. This holds if and only if

$$r^2 + s^2 + 4Rr = (a + c)b + ac = (2s - b)b + ac = 2s\sqrt[3]{4Rrs} - b^2 + ac = 2s\sqrt[3]{4Rrs}$$

or $(r^2 + s^2 + 4Rr)^3 = 32Rrs^4$.

342. Prove that there are infinitely many solutions in positive integers, whose greatest common divisor is equal to 1, of the system

$$\begin{aligned} a + b + c &= x + y \\ a^3 + b^3 + c^3 &= x^3 + y^3 \ . \end{aligned}$$

Solution 1. Suppose that a, b, c are in arithmetic progression, so that $c = 2b - a$ and $x + y = 3b$. Then

$$x^2 - xy + y^2 = \frac{a^3 + b^3 + c^3}{a + b + c} = 3b^2 - 4ab + 2a^2$$

so that

$$3xy = (x + y)^2 - (x^2 - xy + y^2) = 6b^2 + 4ab - 2a^2$$

and

$$xy = 2b^2 + \frac{2a(2b - a)}{3} \ .$$

Therefore

$$(y-x)^2 = (x+y)^2 - 4xy = b^2 - \frac{8a(2b-a)}{3} = \frac{(3b-8a)^2 - 40a^2}{9} \ .$$

Let $p = 3b - 8a$, $q = 2a$. We can get solutions by solving $p^2 - 10q^2 = 9$. Three solutions are $(p,q) = (3,0), (7,2), (13,4)$. The fundamental solution of $u^2 - 10v^2 = 1$ is $(u,v) = (19,6)$. So from any solution $(p,q) = (r,s)$ of $p^2 - 10q^2 = 9$, we get another $(p,q) = (19r + 60s, 6r + 19s)$. For these to yield solutions $(a,b,c;x,y)$ of the original system, we require q to be even and $p + 4q$ to be divisible by 3. Since $19r + 60s \equiv r \pmod 3$ and $6r + 19s \equiv s \pmod 2$, if $(p,q) = (r,s)$ has these properties, then so also does $(p,q) = (19r+60s, 6r+19s)$. Starting with (r,s), we can define integers p and q, and then solve the equations $x + y = 3b$, $y - x = 1$. Since p and so b are odd, these equations have integer solutions. Here are some examples:

$$\begin{aligned}(p,q;a,b,c;x,y) =&(3,0;0,1,2;1,2), (57,18;9,43,77;64,65),\\ &(7,2;1,5,9;7,8), (253,80;40,191,342;286,287) \ .\end{aligned}$$

Solution 2. [D. Dziabenko] Let $a = 3d$. $c = 2b - 3d$, so that $x + y = 3b$ and a, b, c are in arithmetic progression. Then

$$\begin{aligned}a^3 + b^3 + c^3 &= 27d^3 + b^3 + 8b^3 - 36b^2d + 54bd^2 - 27d^3\\ &= 9b^3 - 36b^2d + 54bd^2 = 9b(b^2 - 4b^2d + 6d^2) \ ,\end{aligned}$$

whence $x^2 - xy + y^2 = 3b^2 - 12bd + 18d^2$. Therefore

$$3xy = (x+y)^2 - (a^2 - xy + y^2) = 6b^2 + 12bd - 18d^2$$

so that $xy = 2b^2 + 4bd - 6d^2$ and

$$(x-y)^2 = (x+y)^2 - 4xy = b^2 - 16bd + 24d^2 = (b-8d)^2 - 40d^2 \ .$$

Let $b - 8d = p^2 + 10q^2$ and $d = pq$. Then

$$x - y = \sqrt{p^4 - 20p^2q^2 + 100q^4} = p^2 - 10q^2 \ .$$

Solving this, we find that

$$(a,b,c;x,y) = (3pq, p^2 + 8pq + 10q^2, 2p^2 + 13pq + 20q^2; 2p^2 + 12pq + 10q^2, p^2 + 12pq + 20q^2) \ .$$

Some numerical examples are

$$(a,b,c;p,q) = (3,19,35;24,33), (6,30,54;42,48), (6,57,108;66,105) \ .$$

Any common divisor of a and b must divide $3pq$ and $p^2 + 10q^2$, and so must divide both p and q. [Justify this; you need to be a little careful.] We can get the solutions we want by arranging that the p and q are coprime.

Solution 3. [F. Barekat] Let $m = a + b + c = x + y$ and $n = a^3 + b^3 + c^3 = x^3 + y^3$. Then

$$3xy = m^2 - \frac{n}{m} = \frac{m^3 - n}{m}$$

and

$$\begin{aligned}(x-y)^2 &= \frac{x^3 + y^3}{x+y} - xy = \frac{4n - m^3}{3n}\\ &= \frac{4(a^3 + b^3 + c^3) - (a+b+c)^3}{3(a+b+c)}\\ &= (c-a-b)^2 - \frac{4ab(a+b)}{a+b+c} \ .\end{aligned}$$

Select a, b, c so that
$$\frac{4ab(a+b)}{a+b+c} = 2(c-a-b) - 1 \ .$$
so that $x-y = c-a-b-1$. Then we can solve for rational values of x and y. If we can do this $x+y = a+b+c$ and $x-y = c-a-b-1$. Note that, these two numbers have different parity, so we will obtain fractional values of x and y, whose denominators are 2. However, the equations to be solved are homogeneous, so we can get integral solutions by doubling: $(2a, 2b, 2c; 2x, 2y)$.

Let $c = u(a+b)$. Then
$$4ab = 2(u-1)(u+1)(a+b) - (u+1) \ .$$
Let $u = 4v+3$, Then we get
$$ab = 4(v+1)(2v+1)(a+b) - 4(v+1) \ ,$$
from which
$$[a - 4(v+1)(2v+1)][b - 4(v+1)(2v+1)] = (v+1)[16(v+1)(2v+1)^2 - 1] \ .$$
We use various factorizations of the right side and this equation to determine integer values of a and b, from which the remaining variables c, x and y can be determined.

For example, $v = 0$ yields the equation $(a-4)(b-4) = 15$ from which we get the possibilities
$$(a, b, c) = (5, 19, 72), (7, 9, 48) \ .$$
Doubling to clear fractions, yields the solutions
$$(a, b, c; x, y) = (10, 38, 144; 49, 143), (14, 18, 96; 33, 95) \ .$$
Additional solutions come from $v = 1$:
$$(a, b, c; x, y) = (76, 130, 1442; 207, 1441), (50, 1196, 8722; 1247, 8721) \ .$$

Solution 4. [D. Rhee] An infinite set of solutions is given by the formula
$$\begin{aligned}(a, b, c; x, y) &= (2, n^2+3n, n^2+5n+4; n^2+4n+2, n^2+4n+4) \\ &= (2, n(n+3), (n+1)(n+4); (n+2)^2 - 2, (n+2)^2) \ .\end{aligned}$$
Examples are $(a, b, c; x, y) = (2, 4, 10; 7, 9), (2, 10, 18; 14, 16), (2, 18, 28; 23, 25)$.

Comment. M. Fatehi gave the solution
$$(a, b, c; x, y) = (5, 6, 22; 12, 21) \ .$$

343. A sequence $\{a_n\}$ of integers is defined by
$$a_0 = 0 \ , \quad a_1 = 1 \ , \quad a_n = 2a_{n-1} + a_{n-2}$$
for $n > 1$. Prove that, for each nonnegative integer k, 2^k divides a_n if and only if 2^k divides n.

Solution 1. Let m and n be two nonnegative integers. Then $a_{m+n} = a_m a_{n+1} + a_{m-1}a_n = a_{m+1}a_n + a_m a_{n-1}$. This can be checked for small values of m and n and established by induction. The induction step is
$$\begin{aligned}a_{m+n+1} = 2a_{m+n} + a_{m+n-1} &= 2(a_m a_{n+1} + a_{m-1}a_n) + (a_m a_n + a_{m-1}a_{n-1} \\ &= a_m(2a_{n+1} + a_n) + a_{m-1}(2a_n + a_{n-1}) = a_m a_{n+2} + a_{m-1}a_{n+1} \ .\end{aligned}$$

In particular, for each integer n,

$$a_{2n} = a_n(a_{n-1} + a_{n+1}) \ .$$

It is straightforward to show by induction from the recursion that a_n is odd whenever n is odd and even whenever n is even. Suppose now that n is even. Then $a_{n+1} = 2a_n + a_{n-1} \equiv a_{n-1} \equiv a_1 = 1 \pmod 4$, so that $a_{n-1} + a_{n+1} = 2b_n$ for some odd number b_n. Hence $a_{2n} = 2a_n b_n$. For $k = 0$, we have that $2^k | a_n$ if and only if $2^k | n$. Suppose that this has been established for $k = r$.

Suppose that $n = 2^{r+1}m$ for some integer m. Then $n/2$ is divisible by 2^r, and therefore so is $a_{n/2}$. Hence $a_n = 2a_{n/2}b_{n/2}$ is divisible by 2^{r+1}. On the other hand, suppose that n is not divisible by 2^{r+1}. If n is not divisible by 2^r, then a_n is not so divisible by the induction hypothesis, and so not divisible by 2^{r+1}. On the other hand, if $n = 2^r c$, with c odd, then a_n is divisible by 2^r. But $n/2 = 2^{r-1}c$, so $a_{n/2}$ is not divisible by 2^r. Hence $a_n = 2a_{n/2}b_{n/2}$ is not divisible by 2^{r+1}. The result follows.

Solution 2. For convenience, imagine that the sequence is continued backwards using the recursion $a_{n-2} = a_n - 2a_{n-1}$ for all integer values of the index n. We have for every integer n, $a_{n+1} = 2a_n + a_{n-1} \Rightarrow 2a_{n+1} = 4a_n + 2a_{n-1} \Rightarrow a_{n+2} - a_n = 4a_n + a_n - a_{n-2} \Rightarrow a_{n+2} = 6a_n - a_{n-2}$. Suppose, for some positive integer r, we have established that, for every integer n,

$$a_{n+2^r} = b_r a_n - a_{n-2^r}$$

where $b^r \equiv 2 \pmod 4$. This is true for $r = 1$ with $b_1 = 6$. Then

$$\begin{aligned} b_r a_{n+2^r} &= b_r^2 a_n - b_r a_{n-2^r} \\ \Longrightarrow a_{n+2^{r+1}} + a_n &= b_r^2 a_n - (a_n + a_{n-2^{r+1}}) \\ \Longrightarrow a_{n+2^r} &= b^{r+1} a_n - a_{n-2^{r+1}} \ , \end{aligned}$$

where $b_{r+1} = b_2^2 - 2 \equiv 2 \pmod 4$.

Observe that, since $a_{n+1} \equiv a_{n-1} \pmod 2$ and $a_0 = 0$, $a_1 = 1$, a_n is even if and only if n is even. When n is even, then $a_{n+2} \equiv a_{n-2} \pmod 4$, so that a_n is divisible by 4 if and only if n is.

Let $m \geq 2$ be a positive integer. Suppose that it has been established for $1 \leq s \leq m$, that 2^s divides a_n if and only if 2^s divides n. Then 2^{s+1} will divide a_n only if $n = 2^s p$ for some integer p. Now

$$a_{2^s} = b_{s-1} a_{2^{s-1}} - a_0 = b_{s-1} a_{2^{s-1}} \ ;$$

since $2 \| b^{s-1}$ and $2^{s-1} \| a^{2^{s-1}}$, it follows that $2^s \| a_{2^s}$. (The notation $2^k \| q$ means that 2^k is the highest power of 2 that divides q.) Thus 2^{s+1} does not divide 2^s.

Suppose that it has been established for $1 \leq i \leq p$ that when $n = 2^s i$, $2^{s+1} | n$ if and only if p is even. We have that

$$a_{2^s(p+1)} = b_s 2^s p - a_{2^s(p-1)} \ .$$

If p is even, then $b^s a_{2^s p} \equiv 0 \pmod{2^{s+1}}$, so that $a_{2^s(p+1)} \equiv a_2^s(p-1) \equiv 2^s \pmod{2^{s+1}}$, and $a_{2^s(p+1)}$ is not a multiple of 2^{s+1}. If p is odd, then each term on the right side of the foregoing equation is a multiple of 2^{s+1}, and therefore so is $a^{2^s(p+1)}$. The desired result follows by induction.

Solution 3. The characteristic equation for the recursion is $t^2 - 2t - 1 = 0$, with roots $t = 1 \pm \sqrt{2}$. Solving the recursion, we find that

$$\begin{aligned} a_n &= \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n] \\ &= \frac{1}{\sqrt{2}}\left[\sum_{k=0}^{\infty} \binom{n}{2k+1} 2^k \sqrt{2}\right] \\ &= \sum_{k=0}^{\infty} \binom{n}{2k+1} 2^k = n + \sum_{k=1}^{\infty} \binom{n}{2k+1} 2^k \\ &= n + \sum_{k=1}^{\infty} \frac{n}{2k+1} \binom{n-1}{2k} 2^k \ . \end{aligned}$$

(We use the convention that $\binom{i}{j} = 0$ when $i < j$. Suppose that $n = 2^r s$ where r is a nonnegative integer and s is odd. Since the odd number $2k+1$ divides $n\binom{n-1}{2k} = 2^r s\binom{n-1}{2k}$, $2k+1$ must divide $s\binom{n-1}{2k}$, so that 2^s must divide $n\binom{n-1}{2k} = \binom{n}{2k+1}$. Therefore, 2^{s+1} must divide each term $\binom{n}{2k+1}$ for $k \geq 1$. Therefore $a_n \equiv n \pmod{2^s}$ and the desired result follows.

Comment. Y. Zhao obtained by induction that

$$\begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$$

from which the matrix equation

$$\begin{pmatrix} a_{2n+1} & a_{2n} \\ a_{2n} & a_{2n-1} \end{pmatrix} = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{2n-1} \end{pmatrix}^2$$

yields the equation $a_{2n} = a_n(a_{n-1} + a_{n+1}$.

344. A function f defined on the positive integers is given by

$$f(1) = 1\ , \quad f(3) = 3\ , \quad f(2n) = f(n)\ ,$$

$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n)\ ,$$

for each positive integer n. Determine, with proof, the number of positive integers no exceeding 2004 for which $f(n) = n$.

Solution. Let $g(n)$ be defined for positive integer n by writing n to base 2 and reversing the digits. Specifically, if $n = \sum_{k=0}^r a_k 2^r$ with each a_k equal to 0 or 1 and $a_r = 1$, then $g(n) = \sum_{k=0}^n a_{r-k}2^k$. We prove that $g(n)$ has the properties ascribed to $f(n)$. It is checked that $g(1) = g(2) = g(3) = 1$. Let $n = a_r 2^r + a_{r-1}2^{r-1} + \cdots + a_1 2 + a_0$. Then $2n = a_r 2^{r+1} + \cdots + a_0 2 + 0$ and $g(2n) = 0 \cdot 2^{r+1} + a_0 2^r + \cdots + a_{r-1}2 + a_r = g(n)$.

Since $4n+1 = a_r 2^{r+2} + a_{r-1}2^{r+1} + \cdots + a_1 2^3 + a_0 2 + 0 \cdot 2 + 1$,

$$\begin{aligned} g(n) + g(4n+1) &= (a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r) + (2^{r+2} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r) \\ &= 2^{r+2} + a_0 2^{r+1} + a_1 r^r + \cdots + a_{r-1}2^2 + a_r 2 = 2(2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r \\ &= 2g(a_2 2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r) = 2g(2n+1)\ . \end{aligned}$$

(This uses the fact that $a2^i + a2^i = a2^{i+1}$.)

Since $4n+3 = a_r 2^{r+2} + a_{r-1}2^{r+1} + \cdots + a_1 2^3 + a_0 2 + 1 \cdot 2 + 1$,

$$\begin{aligned} &2g(n) + g(4n+3) \\ &\quad = (a_0 2^{r+1} + a_1 2^r + a_2 2^{r-1} + \cdots + a_{r-1}2^2 + a_r 2) \\ &\qquad + (2^{r+2} + 2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a^r) \\ &\quad = (2^{r+2} + a_0 2^{r+1} + a_1 2^r + \cdots + a_r 2) + (2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_r) \\ &\quad = 2g(2n+1) + g(2n+1) = 3g(2n+1)\ . \end{aligned}$$

We show by induction that $f(n) = g(n)$ for every positive integer n. This is true for $1 \leq n \leq 4$. Suppose it holds for $1 \leq n \leq 4m$. Then

$$f(4m+1) = 2f(2m+1) - f(m) = 2g(2m+1) - g(m) = g(4m+1)\ ;$$

$$f(4m+2) = f(2m+1) = g(2m+1) = g(4m+2) \ ;$$
$$f(4m+3) = 3f(2m+1) - 2f(m) = 3g(2m+1) - 2g(m) = g(4m+3) \ ;$$
$$f(4m+4) = f(2m+2) = g(2m+2) = f(4m+4) \ .$$

Thus we have a description of $f(n)$.

For $f(n) = n$, it is necessary and sufficient that n is a palindrome when written to base 2. We need to find the number of palindromes between 1 and $2004 = (11111010100)_2$ inclusive. The number of $(2r-1)-$ and $2r-$digit palindromes is each 2^{r-1} as the first and last digits must be 1 and there are $r-1$ other matching pairs of digits or central digits that can be set to either 0 or 1. The number of palindromes up to $2^{11} - 1 = 2047$ is $2(1+2+4+8+16) + 32 = 94$. The only palindromes between 2004 and 2048 are $(11111011111)_2$ and $(11111111111)_2$, and these should not be counted. Therefore, there are exactly 92 palindromes, and therefor 92 solutions of $f(n) = n$ between 1 and 2004, inclusive.

345. Let $\mathcal{C}$ be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of $\mathcal{C}$, keeping the new faces perpendicular to the diagonals of the cuhe and keeping the newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.

Solution 1. In the situation where the cuts pass through the midpoints of the edges, yielding a cube-octahedron with six square and eight equilateral-triangular sides, we find that the square faces have area 2 and the triangular faces have area $(\sqrt{3}/4)(\sqrt{2}) = \sqrt{6}/4 < 2$. Moving the cuts closer to the vertices yields triangular faces of area less than 2 and octahedral faces of area greater than 2. Thus, for equal areas of the corner and face figures, the cuts must be made a a distance exceeding 1 from each vertex.

The corner faces of the final solid are hexagons formed by large equilateral triangles with smaller equilateral triangles clipped off each vertex; the other faces are squares (diamonds) in the middle of the faces of the cube. Let the square faces have side length x. The vertices of this face are distant $1 - (x/\sqrt{2})$ from the edge of the cube, so that smaller equilateral triangles of side $\sqrt{2}(1 - (x/\sqrt{2})) = \sqrt{2} - x$ are clipped off from a larger equilateral triangle of side $2(\sqrt{2} - x) + x = 2\sqrt{2} - x$. The areas of the hexagonal faces of the solid figure are each

$$\frac{\sqrt{3}}{4}[(2\sqrt{2} - x)^2 - 3(\sqrt{2} - x)^2] = \frac{\sqrt{3}}{2} + \frac{x\sqrt{6}}{2} - \frac{x^2\sqrt{3}}{2} \ .$$

For equality, we need

$$x^2 = \frac{\sqrt{3}}{2}[1 + x\sqrt{2} - x^2] \ ,$$

or

$$(2 + \sqrt{3})x^2 - x\sqrt{6} - \sqrt{3} = 0 \ .$$

Hence

$$x = \frac{\sqrt{6} + \sqrt{8\sqrt{3} + 18}}{2(2 + \sqrt{3})}$$

and the common area is

$$x^2 = \frac{6 + 2\sqrt{3} + \sqrt{27 + 12\sqrt{3}}}{7 + 4\sqrt{3}} = (6 + 2\sqrt{3} + \sqrt{27 + 12\sqrt{3}})(7 - 4\sqrt{3}) \ .$$

Solution 2. Let the cut be made distant u from a vartex. As in Solution 1, we argue that $1 < u < 2$. Then the edge of the square face of the final solid is distant $u/\sqrt{2}$ from the vertex of the cube and $\sqrt{2}(1 - (u/2))$ from the centre of the face. Thus, the square face has side length $\sqrt{2}(2 - u)$ and area $8 - 8u + 2u^2$.

The hexagonal face of the solid consists of an equilateral triangle of side $\sqrt{2}u$ with three equilateral triangles of side $\sqrt{2}(u-1)$ clipped off. Its area is $(\sqrt{3}/2)[-2u^2 + 6u - 3$. For equality of area of all the faces, we require that

$$2(8 - 8u + 2u^2) = \sqrt{3}(-2u^2 + 6u - 3)$$

or

$$2(2+\sqrt{3})u^2 - 2(8+3\sqrt{3})u + (16+3\sqrt{3}) = 0 \ .$$

Solving this equation and taking the root less than 2 yields that

$$u = \frac{(8+3\sqrt{3}) - \sqrt{9+4\sqrt{3}}}{2(2+\sqrt{3})} \ ,$$

whence

$$2 - u = \frac{\sqrt{3} + \sqrt{9+4\sqrt{3}}}{2(2+\sqrt{3})} \ .$$

Thus, the common area is

$$2(2-u)^2 = \frac{6+2\sqrt{3}+\sqrt{27+12\sqrt{3}}}{7+4\sqrt{3}} = (6+2\sqrt{3}+\sqrt{27+12\sqrt{3}})(7-4\sqrt{3}) \ .$$