Solutions

311. Given a square with a side length 1, let P be a point in the plane such that the sum of the distances from P to the sides of the square (or their extensions) is equal to 4. Determine the set of all such points P.

Solution. If the square is bounded by the lines $x = 0$, $x = 1$, $y = 0$ and $y = 1$ in the Cartesian plane, then the required locus is equal the octagon whose vertices are $(0, 2)$, $(1, 2)$, $(2, 1)$, $(2, 0)$, $(1, -1)$, $(0, -1)$, $(-1, 0)$, $(-1, 1)$. Any point on the locus must lie outside of the square, as within the square the sum of the distances is equal to 2. If, for example, a point on the locus lies between $x = 0$ and $x = 1$, the sum of the distances to the vertical sides is 1, and it must be 1 unit from the nearer horizontal side and 2 units from the farther horizontal side. If, for example, a point on the locus lies to the left of $x = 0$ and above $y = 1$ and has coordinates (u, v), then

$$|u| + (1 + |u|) + v + (v - 1) = 4$$

or

$$-u + 1 - u + v + v - 1 = 4 \quad \text{or} \quad v - u = 2 \ .$$

Thus it can be shown that every point on the locus lies on the octagon, and conversely, it is straightforward to verify that each point on the octagon lies on the locus.

312. Given ten arbitrary natural numbers. Consider the sum, the product, and the absolute value of the difference calculated for any two of these numbers. At most how many of all these calculated numbers are odd?

Solution. Suppose that there are k odd numbers and $10 - k$ even numbers, where $0 \leq k \leq 10$. There are $k(10 - k)$ odd sums, $k(10 - k)$ odd differences and $\frac{1}{2}k(k - 1)$ odd products (on the presumption that the numbers chosen are distinct), giving a total of

$$\frac{1}{2}(39k - 3k^2) = \frac{3}{2}\left[\left(\frac{13}{2}^2\right) - \left(\frac{13}{2} - k\right)^2\right]$$

odd results. This quantity achieves its maximum when $k = 13/2$, so the maximum number 63 of calculated numbers occurs when $k = 6$ or $k = 7$.

Comment. If we allow a number to be operated with itself, then the maximum occurs when $k = 7$.

313. The three medians of the triangle ABC partition it into six triangles. Given that three of these triangles have equal perimeters, prove that the triangle ABC is equilateral.

Solution. [P. Shi] Let a, b, c be the respective lengths of the sides BC, CA, AB, and u, v, w the respective lengths of the medians AP, BQ, CR (P, Q, R the respective midpoints of BC, CA, AB). If G is the centroid of the triangle ABC, then

$$AG : GP = BG : GQ = CG : GR = 2 : 1 \ .$$

We need three preliminary lemmata.

Lemma 1. $4u^2 = 2b^2 + 2c^2 - a^2; 4v^2 = 2c^2 + 2a^2 - b^2; 4w^2 = 2a^2 + 2b^2 - c^2$.

Proof. This can be established, either by representing the medians vertorially in terms of the sides and applying the cosine law to the whole triangle, or by applying the cosine law to pairs of inner triangles along an edge. ♠

Lemma 2. $u < v; u = v; u > v$ according as $a > b; a = b; a < b$, with analogous results for other pairs of medians and sides.

Proof. $4(u^2 - v^2) = 3(b^2 - a^2)$. ♠

Lemma 3. If triangle BCR and BCQ have the same perimeter, then $b = c$.

Proof. Equality of the perimeters is equivalent to $BR+RC = BQ+QC$, so that Q and R are points on an ellipse with foci B and C. Since RQ is parallel to the major axis containing BC, R and Q are reflections of each other in the minor axis, so that $RB = QC$. Hence $b = c$. ♠

We now establish conditions under which the triangle must be isosceles.

Lemma 4. Suppose that two adjacent inner triangles along the same side of triangle ABC have the same perimeter. Then triangle ABC is isosceles.

Solution. For example, the equality of the perimeters of BPG and CPG is equivalent to

$$\frac{1}{2}a + \frac{1}{3}u + \frac{2}{3}v = \frac{1}{2}a + \frac{1}{3}u + \frac{2}{3}w$$

$$\Leftrightarrow v = w \Leftrightarrow b = c\ .♠$$

Lemma 5. Suppose that two adjacent inner triangles sharing a vertex of triangle ABC have the same perimeter. Then triangle ABC is isosceles.

Proof. For example, suppose that triangle BRG and PGB have the same perimeter. Produce CR to point T so that $GR = RT$. Thus, BR is a median of triangle BGT. Produce AP to point S so that $GP = PS$. Thus, CP is a median of triangle CPS.

Since GP joins midpoints of two sides of triangle CTB, $TB \| GP$ and $TB = 2GP = GS$. Since triangle PGB and PSC are congruent (SAS), $BG = SC$. Also, $TG = 2RG = GC$. Hence, triangles TBG and GSC are congruent (SSS).

Let X be the midpoint of GC. A translation that takes T to G takes triangle TBG to triangle GSC and median BR to median SX. We have that

$$\text{Perimeter}(SXC) = \text{Perimeter}(BRG) = \text{Perimeter}(PGB) = \text{Perimeter}(PSC)\ .$$

Applying Lemma 3, we deduce that $GS = GC$, whence $u = w$ and $a = c$. ♠

Lemma 6. If two opposite triangles (say, BGP and AQG) have equal perimeters, then triangle ABC is isosceles.

Proof. The equality of the perimeters of BGP and AQG implies that

$$\frac{1}{2}a + \frac{1}{3}u + \frac{2}{3}v = \frac{1}{2}b + \frac{1}{3}v + \frac{2}{3}u$$

$$\Leftrightarrow 3(a-b) = 2(u-v)\ .$$

By Lemma 2, the latter equation holds if and only if $a = b$. ♠

Let us return to the problem. There are essentially four different cases for the three inner triangles with equal perimeters.

Case 1. The three are adjacent (say BRG, BPG, CPG). Then by Lemmata 4 and 5, $a = b = c$.

Case 2. Two are adjacent along a side and the third is opposite (say BPG, CPG, AQG). Then, by Lemmata 4 and 6, $a = b = c$.

Case 3. Two are adjacent at a vertex and the third is opposite (say BPG, CQG, AQG.) Then, by Lemmata 5 and 6, $a = b = c$.

Case 4. No two are adjacent (say BPG, CQG, ARG). Then we have

$$\frac{1}{2}a + \frac{1}{3}u + \frac{2}{3}v = \frac{1}{2}b + \frac{1}{3}v + \frac{2}{3}w = \frac{1}{2}c + \frac{1}{3}w + \frac{2}{3}v\ .$$

Thus

$$3(a-b) = 2(w-u) + 2(w-v)\ . \qquad (1)$$

Similarly,

$$3(b-c) = 2(u-v) + 2(u-w) \ ; \qquad (2)$$

$$3(c-a) = 2(v-u) + 2(v-w) \ . \qquad (3)$$

Suppose, wolog, that $a \geq b \geq c$. Then $u \leq v \leq w$, so that, by (2), $3(b-c) \leq 0 \Rightarrow b = c \Rightarrow v = w$. But then, by (3), $3(c-a) = 2(v-u) \geq 0 \Rightarrow c = a$.

The result follows.

314. For the real numbers a, b and c, it is known that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = 1 \ ,$$

and

$$a + b + c = 1 \ .$$

Find the value of the expression

$$M = \frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \ .$$

Solution 1. Putting the first equation over a common denominator and using the second equation yields that

$$a + b + c = abc = 1$$

whence

$$\begin{aligned} M &= \frac{1}{1+a+(1/c)} + \frac{1}{1+b+(1/a)} + \frac{1}{1+c+(1/b)} \\ &= \frac{c}{1+c+ac} + \frac{a}{1+a+ab} + \frac{b}{1+b+bc} \\ &= \frac{c}{1+c+(1/b)} + \frac{a}{1+a+(1/c)} + \frac{b}{1+b+(1/a)} \\ &= \frac{bc}{1+b+bc} + \frac{ac}{1+c+ac} + \frac{ab}{1+a+ab} \ . \end{aligned}$$

This yields three different expressions for M over denominators of the form $1 + a + ab$, which when added together yield $3M = 3$ or $M = 1$.

Solution 2. [V. Krakovna] It is clear that $abc \neq 0$ for the expressions to be defined. As before, $abc = 1$, and

$$\frac{1}{1+a+ab} = \frac{1}{1+a+(1/c)} = \frac{c}{c+ca+1} \ .$$

Hence

$$\begin{aligned} M &= \frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \\ &= \frac{c}{c+ca+1} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \\ &= \frac{c+1}{1+c+ca} + \frac{1}{1+b+bc} \\ &= \frac{b(c+1)}{b+bc+1} + \frac{1}{1+b+bc} = \frac{bc+b+1}{b+bc+1} = 1 \ . \end{aligned}$$

315. The natural numbers 3945, 4686 and 5598 have the same remainder when divided by a natural number x. What is the sum of the number x and this remainder?

Solution. Observe that $5598 - 4686 = 912 = 16 \times 57$ and $4686 - 3945 = 741 = 13 \times 57$, so that if a divisor leaves equal remainders for the three numbers, the divisor must also divide evenly into 57. Since $5598 = 98 \times 57 + 12$, $4686 = 82 \times 57 + 12$ and $3945 = 69 \times 57 + 12$. the number x must be 1, 3, 19 or 57. The sums of the number and the remainder are respectively 1, 3, 31 and 69.

316. Solve the equation

$$|x^2 - 3x + 2| + |x^2 + 2x - 3| = 11 \ .$$

Solution. The equation can be rewritten

$$|x - 1|[|x - 2| + |x + 3|] = 11 \ .$$

When $x \leq -3$, the equation is equivalent to

$$2x^2 - x - 12 = 0$$

neither of whose solutions satisfies $x \leq -3$. When $-3 \leq x \leq 1$, the equation is equivalent to $-5x + 5 = 11$ and we get the solution $x = -6/5$. When $1 < x < 2$, the equation is equivalent to $5x - 5 = 11$ which has no solution with $1 < x < 2$. Finally, when $2 \leq x$, the equation is equivalent to $2x^2 - x - 12 = 0$ and we obtain the solution $x = \frac{1}{4}(1 + \sqrt{97})$.

Thus the solutions are $x = -6/5, (1 + \sqrt{97})/4$.

317. Let $P(x)$ be the polynomial

$$P(x) = x^{15} - 2004x^{14} + 2004x^{13} - \cdots - 2004x^2 + 2004x \ ,$$

Calculate $P(2003)$.

Solution 1. For each nonnegative integer n, we have that

$$x^{n+2} - 2004x^{n+1} + 2003x^n = x^n(x - 1)(x - 2003) \ .$$

Therefore,

$$\begin{aligned} P(x) &= (x^{15} - 2004x^{14} + 2003x^{13}) + (x^{13} - 2004x^{12} + x^{11}) + \cdots + (x^3 - 2004x^2 + 2003x) + x \\ &= (x^{13} + x^{11} + \cdots + x)(x - 1)(x - 2003) + x \quad , \end{aligned}$$

whereupon $P(2003) = 2003$.

Solution 2. [R. Tseng]

$$P(x) = (x^{15} - 2003x^{14}) - (x^{14} - 2003x^{13}) + (x^{13} - 2003x^{12}) - \cdots - (x^2 - 2003x) + x$$

whence $P(2003) = 0 - 0 + 0 - \cdots - 0 + 2003 = 2003$.