Solutions.

290. The School of Architecture in the *Olymon* University proposed two projects for the new Housing Campus of the University. In each project, the campus is designed to have several identical dormitory buildings, with the same number of one-bedroom apartments in each building. In the first project, there are 12096 apartments in total. There are eight more buildings in the second project than in the first, and each building has more apartments, which raises the total of apartments in the project to 23625. How many buildings does the second project require?

Solution. Let the number of buildings in the first project be n. Then there must be $12096/n$ apartments in each of them. The number of buildings in the second project is $n+8$ with $23625/(n+8)$ apartments in each of them. Since the number of apartments is an integer, $n+8$, and so n, is odd. Furthermore, $12096 = 2^6 \cdot 3^3 \cdot 7$ and $23625 = 3^3 \cdot 5^3 \cdot 7$. Since n is an odd factor of 12096, n must take one of the values 1, 3, 7, 9, 21, 27, 63 or 189. Since $n+8$ must be a factor of 23625, the only possible values for n are 1, 7 or 27. Taking into account that the number of apartments in each building of the second project is more than the number of apartments in each building of the first project, n must satisfy the inequality

$$\frac{12096}{n} < \frac{23625}{n+8} \ ,$$

which is equivalent to $n > 512/61$. Thus, $n \geq 9$. Therefore, $n = 27$ and $n+8 = 35$. The second project requires 35 buildings.

291. The n-sided polygon $A_1, A_2, \cdots, A_n$ $(n \geq 4)$ has the following property: The diagonals from each of its vertices divide the respective angle of the polygon into $n-2$ equal angles. Find all natural numbers n for which this implies that the polygon $A_1A_2\cdots A_n$ is regular.

Solution. Let the measures of the angles of the polygon at each vertex A_i be a_i for $1 \leq i \leq n$. When $n = 4$, the polygon need not be regular. Any nonsquare rhombus has the property.

Let n exceed 4. Consider triangle $A_1A_2A_n$. We have that

$$a_1 + \frac{a_2}{n-2} + \frac{a_n}{n-2} = 180^\circ \ .$$

Since the sum of the exterior angles of an $n-$gon is $a_1 + a_2 + \cdots + a_n = (n-2)180^\circ$, we find that

$$(n-3)a_1 = a_3 + a_4 + \cdots + a_{n-1} \ .$$

Suppose, wolog, that a_1 is a largest angle in the polygon. Then

$$(n-3)a_1 = a_3 + a_4 + \cdots + a_{n-1} \leq (n-3)a_1$$

with equality if and only if $a_1 = a_3 = \cdots = a_{n-1}$. Suppose, if possible that one of a_2 and a_n is strictly less than a_1. We have an inequality for a_4 analogous to the one given for a_1, and find that

$$(n-3)a_4 = a_1 + a_2 + a_6 + \cdots + a_n < (n-3)a_1 = (n-3)a_4 \ ,$$

a contradiction. Hence, all the angles a_i must be equal and the polygon regular.

292. 1200 different points are randomly chosen on the circumference of a circle with centre O. Prove that it is possible to find two points on the circumference, M and N, so that:

- M and N are different from the chosen 1200 points;
- $\angle MON = 30^\circ$;
- there are *exactly* 100 of the 1200 points inside the angle MON.

Solution. The existence of the points M and N will be evident when we prove that there is a central angle of $30°$ which contains exactly 100 of the given points. Construct six diameters of the circle for which none of their endpoints coincide with any of the given points and they divide its circumference into twelve equal arcs of $30°$; such a construction is always possible. If one of the angles contains exactly 100 points, then we have accomplished our task. Assume none of the angles contains 100 points. Then some contain more and others, less. Wolog, we can choose adjacent angles for which the first contains $d_1 > 100$ points and the second $d_2 < 100$ points. Define a function d which represents the number of points inside a rotating angle with respect to its position, and imagine this rotating angle moves from the position of the first of the adjacent angles to the second. As the angle rotates, one of the following occurs: (i) a new point enters the rotating angle; (ii) a point leaves the rotating angle; (iii) no point leaves or enters; (iv) one point leaves while another enters. Thus, the value of d changes by 1 at a time from d_1 to d_2, as so at some point must take the value 100. The desired result follows.

293. Two players, Amanda and Brenda, play the following game: Given a number n, Amanda writes n different natural numbers. Then, Brenda is allowed to erase several (including none, but not all) of them, and to write either $+$ or $-$ in front of each of the remaining numbers, making them positive or negative, respectively, Then they calculate their sum. Brenda wins the game is the sum is a multiple of 2004. Otherwise the winner is Amanda. Determine which one of them has a winning strategy, for the different choices of n. Indicate your reasoning and describe the strategy.

Solution. Amanda has a winning strategy, if $n \leq 10$. She writes the numbers $1, 2, 2^2, 2^3, \cdots, 2^{n-1}$. Recall that, for any natural number k, $2^k > 1 + 2 + \cdots + 2^{k-1}$. Since $2^{10} = 1024$, it is clear that, regardless of Brenda's choice, the result sum but lies between -1023 and 1023, inclusive, and it is not 0, since its sign coincides with the sign of the largest participating number. Hence, the sum cannot be a multiple of 2004.

Let $n \geq 11$. Then it is Brenda that has a winning strategy. The set C of numbers chosen by Amanda has $2^n - 1 > 2003$ different non-empty subsets. By the Pigeonhole Principle, two of these sums must leave the same remainder upon division by 2004. Let A and B be two sets with the same remainder. Brenda assigns a positive sign to all numbers that lie in A but not in B; she assigns a negative sign to all numbers that lie in B but not in A; she erases all the remaining numbers of C. The sum of the numbers remaining is equal to the difference of the sum of the numbers in A and B, which is divisible by 2004. Thus, Brenda wins.

Therefore, Amanda has a winning strategy when $n \leq 10$ and Brenda has a winning strategy when $n \geq 11$.

294. The number $N = 10101 \cdots 0101$ is written using $n+1$ ones and n zeros. What is the least possible value of n for which the number N is a multiple of 9999?

Solution. Observe that $N = 1 + 10^2 + \cdots + 10^{2n}$, $9999 = 3^2 \cdot 11 \cdot 101$ and $10^2 \equiv 1$ modulo 9 and modulo 11. Modulo 9 or modulo 11, $N \equiv n + 1$, so that N is a multiple of 99 if and only if $n \equiv 98 \pmod{99}$. N is a multiple of 101 if and only if n is odd. Hence the smallest value of n for which N is a multiple of 9999 is 197.

295. In a triangle ABC, the angle bisectors AM and CK (with M and K on BC and AB respectively) intersect at the point O. It is known that

$$|AO| \div |OM| = \frac{\sqrt{6} + \sqrt{3} + 1}{2}$$

and

$$|CO| \div |OK| = \frac{\sqrt{2}}{\sqrt{3} - 1} .$$

Find the measures of the angles in triangle ABC.

Solution. Let $AB = c$, $BC = a$, $AC = b$, $CM = x$, $AK = y$, $\angle ABC = \beta$, $\angle ACB = \gamma$ and $\angle CAB = \alpha$. In triangle AMC, CO is an angle bisector, whence

$$AC : CM = AO : OM \Longleftrightarrow \frac{b}{x} = \frac{\sqrt{6}+\sqrt{3}+1}{2} \ . \tag{1}$$

In triangle ABC, AM is an angle bisector, whence

$$AB : AC = BM : CM \Longleftrightarrow \frac{c}{b} = \frac{a-x}{x} \Longleftrightarrow \frac{x}{a} = \frac{b}{c+b} \ . \tag{2}$$

Multiplying (1) and (2), we get

$$\frac{b}{a} = \frac{(\sqrt{6}+\sqrt{3}+1)b}{2(c+b)} \Longleftrightarrow a = \frac{2(b+c)}{\sqrt{6}+\sqrt{3}+1} \ . \tag{3}$$

Similarly, in triangle AKC, AO is an angle bisector. Hence

$$CO : OK = AC : AK \Longleftrightarrow \frac{\sqrt{3}-1}{\sqrt{2}} = \frac{y}{b} \ . \tag{4}$$

In triangle ABC, CK is an angle bisector. Hence

$$BC : AC = BK : AK \Longleftrightarrow \frac{a}{b} = \frac{c-y}{y} \Longleftrightarrow \frac{a+b}{b} = \frac{c}{y} \ . \tag{5}$$

Multiplying (4) and (5), we get

$$\frac{c}{b} = \frac{(a+b)(\sqrt{3}-1)}{b\sqrt{2}} \Longleftrightarrow c = \frac{(a+b)(\sqrt{3}-1)}{\sqrt{2}} \ . \tag{6}$$

Solve (3) and (6) to get b and c in terms of a. We find that

$$(b,c) = \left(\left(\frac{\sqrt{3}+1}{2} \right) a, \left(\frac{\sqrt{6}}{2} \right) a \right) \ .$$

From the Law of Cosines for triangle ABC,

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab} = \frac{1}{2} \Longleftrightarrow \gamma = 60^\circ \ .$$

From the Law of Sines, we have that

$$\frac{\sin\alpha}{\sin\gamma} = \frac{a}{c} \Longleftrightarrow \sin\alpha = \frac{1}{\sqrt{2}} \Longleftrightarrow \alpha = 45^\circ \ .$$

The remaining angle $\beta = 75^\circ$.

296. Solve the equation

$$5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x} \ .$$

Solution 1. [G. Siu; T. Liu] Let $u = \sin x$. For the equation to be meaningful, we require that $u \neq 0$ (mod π). The equation is equivalent to

$$\begin{aligned} 0 &= 4u^4 - 10u^3 + 10u^2 - 5u + 1 = (u-1)(4u^3 - 6u^2 + 4u - 1) \\ &= (u-1)[u^4 - (1-u)^4] = (u-1)[u^2 - (u-1)^2][u^2 + (u-1)]^2 \\ &= u(u-1)(2u-1)[u^2 + (u-1)^2] \ . \end{aligned}$$

We must have that $u = 1$ or $u = 1/2$, whence $x \equiv \pi/6, \pi/2, 5\pi/6 \pmod{2\pi}$.

Solution 2. We exclude $x \equiv 0 \pmod{\pi}$. Let $y = \sin x + (2\sin x)^{-1}$. The given equation is equivalent to

$$0 = 2y^2 - 5y + 3 = (2y - 3)(y - 1) \ .$$

Thus

$$\sin x + \frac{1}{2\sin x} = 1$$

or

$$\sin x + \frac{1}{2\sin x} = \frac{3}{2} \ .$$

The first equation leads to

$$0 = \sin^2 x + (\sin x - 1)^2$$

with no real solutions, while the second leads to

$$0 = 2\sin^2 x - 3\sin x + 1 = (2\sin x - 1)(\sin x - 1) \ ,$$

whence it follows that $x \equiv \pi/6, \pi/2, 5\pi/6 \pmod{2\pi}$.