Solutions

269. Prove that the number

$$N = 2 \times 4 \times 6 \times \cdots \times 2000 \times 2002 + 1 \times 3 \times 5 \times \cdots \times 1999 \times 2001$$

is divisible by 2003.

Solution 1. We will start with more general observations. Let k be a natural number, $A = 2 \times 4 \times 6 \times \cdots \times (2k)$, $B = 1 \times 3 \times 5 \times \cdots \times (2k-1)$, $C = 2k+1$ and $M = A + B$. Since $1 = C - 2k$, $3 = C - (2k-2)$ and so on, $B = (C-2k)(C-(2k-2)) \cdots (C-2)$. Upon expansion, we find that the only term in the right side that does not contain C is $(-1)^k \times 2 \times 4 \times \cdots \times (2k)$. Thus

$$M = C \times \text{ natural number} + (1 + (-1)^k) \times A \ ,$$

so that, when k is odd (for example, when $k = 1001$), M is divisible by C. The result follows.

Solution 2. [T. Yue] Modulo 2003,

$$\begin{aligned} &2 \times 4 \times 6 \times \cdots \times 2000 \times 2002 \\ &\qquad \equiv (-2001) \times (-1999) \times (-1997) \times \cdots \times 3 \times 1 \\ &\qquad = -(2001 \times 1999 \times 1997 \times \cdots \times 3 \times 1) \ . \end{aligned}$$

Therefore, $N \equiv 0 \pmod{2003}$, *i.e.*, N is divisible by 2003.

270. A straight line cuts an acute triangle into two parts (not necessarily triangles). In the same way, two other lines cut each of these two parts into two parts. These steps repeat until all the parts are triangles. Is it possible for all the resulting triangle to be obtuse? (Provide reasoning to support your answer.)

Solution 1. It is clear that if in the final step there are k cuts, made as required, they form $k+1$ triangles. Assume, if possible, that all of these triangles be obtuse. Note the total number of acute or right angles in the configuration after each cut. When the cutting line intersects an existing side of a triangle, it forms two new angles with a sum of $180°$, so that at least one of them is acute or right. When the cutting line passes through a vertex of a triangle, it forms two new angles, dividing the existing angle (smaller than $180°$) into smaller angles, so that there is one more acute or right angle than before. Hence at each step, the total number of acute and right angles in the configuration increases at least by 2. Starting from a configuration with three such angles, after k steps, we get at least $2k+3$ acute or right angles. On the other hand, in $k+1$ obtuse triangles, there must be exactly $2(k+1)$ non-obtuse angles. This contradicts our assumption, so that the answer to the question of the problem is “no”.

Solution 2. Suppose that there were a way to cut the given triangle into t obtuse triangles. According to the required procedure of cutting, if two triangles with a common vertex appear after one cut, then they will lie on the same side of the plane with respect to another line segment (say, a side of the triangle or a previous cut). Denote by n the number of points that are vertices of the obtuse triangles but not vertices of the given triangle. On the one hand, the sum of the interior angles in all the triangles is $180t°$. On the other hand, for each of the n points, the sum of all triangular angles at a vertex there is $180°$. So the sum of all the interior angles of the triangles will be $(180n + 180)°$ (we must add the sum of the angles of the original triangle). Hence $t = n + 1$. However, only the n interior vertices can be vertices of an obtuse angle, and each of them can be the vertex of at most one obtuse angle. Hence $t \le n$, yielding a contradiction. Thus, it is impossible to cut the original triangle into obtuse triangles only.

271. Let x, y, z be natural numbers, such that the number

$$\frac{x - y\sqrt{2003}}{y - z\sqrt{2003}}$$

is rational. Prove that

(a) $xz = y^2$;

(b) when $y \neq 1$, the numbers $x^2 + y^2 + z^2$ and $x^2 + 4z^2$ are composite.

Solution. (a) Since the given number is rational, it can be represented as a reduced fraction p/q, where p and $q \neq 0$ are two coprime integers. This yields

$$xq - yp = (yq - zp)\sqrt{2003} \ .$$

Since the left side is rational, the right must be as well. Since $\sqrt{2003}$ is irrational, both sides must vanish. Thus $xq - yp = yq - zp = 0$, whence $x/y = y/z = p/q$, so that $xz = y^2$.

(b) Let $M = x^2 + y^2 + z^2$ and $N = x^2 + 4z^2$. We will prove that M and N are both composite, provided that $y \neq 1$. Since $xz - y^2$,

$$M = x^2 + y^2 + z^2 = x^2 + 2xz + z^2 - y^2 = (x + z)^2 - y^2 = (x + z - y)(x + z + y) \ .$$

For M to be composite, the smaller factor, $x + z - y$ must differ from 1. (It cannot equal -1. Why?) Since y is a natural number distinct from 1, $y > 1$. As $xz = y^2$, at least one of x and z is not less than y. Say that $x \geq y$. If $x = y$, then $z = y$ and $x + z - y = y > 1$; if $x > y$, then $x + z - y \geq z > 1$. Thus in all possible cases, $x + y - z > 1$ and M is the product of two natural numbers exceeding 1.

Similarly,

$$N = x^2 + 4z^2 = x^2 + 4xz + 4z^2 - 4y^2 = (x + 2z)^2 - (2y)^2 = (x + 2z - 2y)(x + 2z + 2y) \ .$$

To prove that N is composite, it suffices to show that the smaller factor $x + 2z - 2y$ exceeds 1. (Why cannot this factor equal -1?) We prove this by contradiction. Suppose, if possible, that $x + 2z - 2y = 1$. Then $x + 2z = 2y + 1$, whence

$$x^2 + 4xz + 4z^2 = 4y^2 + 4y + 1 \Leftrightarrow x^2 + 4z^2 = 4y + 1 \ .$$

However, it is clear that $x^2 + 4z^2 \geq 4xz = 4y^2$, from which it follows that $4y + 1 \geq 4y^2$. But this inequality is impossible when $y > 1$. Thus, we conclude that $x + 2z - 2y \neq 1$ and so N is composite.

272. Let $ABCD$ be a parallelogram whose area is 2003 sq. cm. Several points are chosen on the sides of the parallelogram.

(a) If there are 1000 points in addition to A, B, C, D, prove that there always exist three points among these 1004 points that are vertices of a triangle whose area is less that 2 sq. cm.

(b) If there are 2000 points in addition to A, B, C, D, is it true that there always exist three points among these 2004 points that are vertices of a triangle whose area is less than 1 sq. cm?

Solution. (a) Since there are 1000 points on the sides of a parallelogram, there must be at least 500 points on one pair of adjacent sides, regardless of the choice of points. Wolog, let these points be on the sides AB and BC of the parallelogram. and let m of the points $P_1, P_2, \cdots, P_m$ be on AB and k of the points $Q_1, Q_2, \cdots, Q_k$ be on BC. Let P_1 and Q_1 be the points closest to B. Connect the vertex C to $P_1, P_2, \cdots, P_m$ and the point P_1 to $Q_1, Q_2, \cdots, Q_k$ to get $m + k + 1$ triangles the sum of whose areas equals the area of ABC. Thus $[ABC] = \frac{1}{2}[ABCD] = 1001.5$ sq cm. Let us assume that each of these $m + k + 1$ triangles has an area that exceeds 2 sq cm. Then $[ABC] \geq 501 \times 2 = 1002 > 1001.5$, a contradiction. Therefore, at least one of these triangles must have an area of less than 2 sq cm.

(b) No, this is not always true. We will construct a counterexample to justify this answer. Let us choose 2000 points on the sides of $ABCD$ so that 1000 of them are on AB and 1000 of them are on CD. We will consider the first set of 1000 points, and then do symmetrical constructions and considerations for the second set. Using the notation from (a), let $m = 1000, k = 0$ and select the points so that $BP_1 = P_1P_2 = P_2P_3 =$

$\cdots = P_{1000}A$. Then the triangle CBP_1, CP_1P_2, $\cdots$, $CP_{1000}A$ have the same area, say s sq cm. However, $s = [ABC]/1001 = (1001.5)/(1000) > 1$; thus, this choice of the first 1000 points allows a construction of triangles such that the area of each of them exceeds 1 sq cm. Similarly, all triangles formed symmetrically with vertices among the other set of 1000 points have an area which exceeds 1 sq cm. So it is not true that there always exists three points among the chosen 2000 points and the points A, B, C, D that are vertices of a triangle whose area is less than 1 sq cm.

Comments. (1) It was not specified in the text of the question that the three points chosen to be the vertices of a triangle have to be non-collinear. Otherwise, we get the trivial case of a "triangle" with an area of 0, which is not interesting, because $0 < 1$, $0 < 2$ and such a triangle will be an example of existence in both cases. However, it is expected that candidates will make a reasonable interpretation of the problem that renders it nontrivial.

(2) Looking into possible interpretations of this problem, Michael Lipnowski came up with a different, but very similar, and interesting problem. *Let $ABCD$ be a parallelogram whose area is 2003 sq cm. Several points are chosen inside the parallelogram. (a) If there are 1000 points in addition to A, B, C, D, prove that there are always three points among these 1004 points that are vertices of a triangle whose area is less than 2 sq cm. (b) If there are 2000 points in addition to A, B, C, D, is it true that there are always three points among the 2004 points that are vertices of a triangle whose area is less than 1 sq cm.* We provide a solution to this problem. Please note that the answer to (b) differs from the answer of the corresponding part of the original question.

(a) Let $|AB| = x$, $|AD| = y$. Let P and Q lie on AB and CD respectively, so that $PQ \| AD$ and $|AP| = |DQ| = (4/2003)x$. This way, we have a parallelogram "cut" from $ABCD$. Construct analogous parallelograms with respect to the sides AB, BC and CD, drawing lines parallel to these sides, so that each of them has a width of $(4x)/2003$ or $(4y)/2003$ respectively. (1) If at least one of the points lies within the parallelograms "cut", say, R, is within $APQD$, then $[ARD] < (1/2)(4/2003)[ABCD] = (1/2)(4/2003)(2003) = 2$, so this proves what is required. (2) Let us assume that all 1000 points (without the vertices of course) lie within the interior parallelogram $KLMN$ whose vertices are the intersection points of the four lines drawn before. Clearly, it is similar to $ABCD$, and the coefficient of proportionality is 1995/2003, so its area is $(1995/2003)^2 \cdot (2003) = (1995^2)/(2003)$. Divide $KLMN$ into 499 congruent parallelograms (for example, by drawing 498 equally spaced lines parallel to KL). Then, since $1000 = 2 \times 499 + 2$ points lie inside $KLMN$, at least one of the 499 parallelograms contains at least three of them, according to the extended pigeonhole principle. Consider the triangle formed by them. Since each of these parallelograms has an area equal to $(1/499)[KLMN] = (1/499)(1995^2/2003) < (1995 \cdot 1996)/(499 \cdot 2003) = 4 \cdot (1995/2003) < 4$, then the area of the triangle will not exceed half of 4, namely 2. So there must be at least one triangle inside $ABCD$ of area less than 2.

(b) Yes, it is always true that there exists three among the 2004 points that are vertices of a triangle with area less than 1. Proceed as in (a) except for the following differences: (1) Construct the parallel lines so that the width of the "cut" parallelograms is $(2x)/2003$ or $(2y)/2003$, respectively. Now, the parallelogram $KLMN$ is similar to $ABCD$, with a coefficient of proportionality 1999/2003 and an area of $(1999^2)/2003$. (2) Divide $KLMN$ into 999 congruent parallelograms. Since $2000 = 2 \times 999 + 2$ points lie within 999 regions, at least one region contains at least three of the points. Similar calculations show that in this case, the area of the triangle formed by these three points has area less than 1. The result holds.

273. Solve the logarithmic inequality

$$\log_4(9^x - 3^x - 1) \geq \log_2 \sqrt{5} \ .$$

Solution. Let $3^x = y$. Then $y > 0$ and the given inequality is equivalent to $\log_4(y^2 - y - 1) \geq \log_2 \sqrt{5}$. Since the logarithmic function is defined only for positive numbers, we must have $y^2 - y - 1 > 0$. In this domain, the inequality is equivalent to $y^2 - y - 1 \geq 5$ or $(y+2)(y-3) \geq 0$. The solution of the last inequality consists of all numbers not less than 3 (since $y > 0$). Hence $3^x \geq 3$ or $x \geq 1$. Thus, the inequality is satisfied if and only if $x \geq 1$.

Comment. It is very important before starting to solve the inequality to determine the domains so that all functions are well-defined. It is mandatory to take these restrictions into consideration for the final answer as well as along the way in making transformations.

274. The inscribed circle of an isosceles triangle ABC is tangent to the side AB at the point T and bisects the segment CT. If $CT = 6\sqrt{2}$, find the sides of the triangle.

Solution. Denote the midpoint of CT by K, and the tangent point of the inscribed circle and BC by L. Then, from the given information,

$$CK = \frac{1}{2}CT \ . \tag{1}$$

We will use the standard notation a, b, c for the lengths of BC, CA and AB, respectively. It is not specified which two sides of the isosceles triangle are equal, so there are two possible cases.

Case 1. $AC = BC$ or $a = b$. Then T is also the midpoint of AB. By the tangent-secant theorem, $CL^2 = CK \cdot CT$, which together with (1) implies that $(a - (c/2))^2 = CL^2 = (1/2)CT^2 = 36$. Hence $a = 6 + (c/2)$ (2).

On the other hand, from the Pythagorean theorem applied to triangle BCT, we get that $a^2 = (c^2/4) + 72$. Using (2), we obtain that

$$\left(6 + \frac{c}{2}\right)^2 = \frac{c^2}{4} + 72 \Leftrightarrow 36 + 6c = 72 \Leftrightarrow c = 6 \ ,$$

whence $a = b = 9$. So the lengths of the triangle are $(a, b, c) = (9, 9, 6)$.

Case 2. $AB = AC$ or $c = b$. Now L is the midpoint of the side BC so that

$$CL^2 = CK \cdot CT = (1/2)CT^2 \Leftrightarrow (a^2/4) = (1/2)(6\sqrt{2})^2 = 36 \Leftrightarrow a = 12 \ . \tag{3}$$

Next we have to calculate the lengths of AB and AC. From the cosine law, applied to triangle BCT with $\beta = \angle ABC$,

$$\begin{aligned} CT^2 &= BT^2 + BC^2 - 2BT \cdot BC \cos\beta \\ &\Leftrightarrow (6\sqrt{2})^2 = (a^2/4) + a^2 - a^2 \cos\beta \\ &\Leftrightarrow 72 = 36 + 144 - 144\cos\beta \Leftrightarrow \cos\beta = 3/4 \ . \end{aligned}$$

On the other hand, the cosine law for triangle ABC leads to

$$\begin{aligned} b^2 &= c^2 + a^2 - 2ca\cos\beta = b^2 + a^2 - 2ba\cos\beta \\ &\Leftrightarrow \cos\beta = a/2b \ . \end{aligned}$$

This, with (3), implies that $c = b = 8$. Therefore, $(a, b, c) = (8, 8, 12)$.

275. Find all solutions of the trigonometric equation

$$\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x \ .$$

Solution 1. [M. Lipnowski] Note that, if $x = \theta$ satisfies the equation, then so does $x = \theta + \pi$. Thus, it suffices to consider $0 \le x \le \pi$. A simple computation shows that $x = \pi/2$ is not a solution, so that we may assume that $\cos x \neq 0$. Multiplying both sides of the equation by $2\cos x \neq 0$ yields that

$$\begin{aligned} &\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x \\ &\quad \Leftrightarrow 2\sin x\cos x - 2\sin 3x\cos x + 2\sin 5x\cos x = 2\cos^2 x - 2\cos 3x\cos x + 2\cos 5x\cos x \\ &\quad \Leftrightarrow \sin 2x - (\sin 4x + \sin 2x) + \sin 6x + \sin 4x) = 1 + \cos 2x - (\cos 4x + \cos 2x) + (\cos 6x + \cos 4x) \\ &\quad \Leftrightarrow \sin 6x - \cos 6x = 1 \ . \end{aligned}$$

Squaring both sides of the last equation, we get

$$\sin^2 6x - 2\sin 6x\cos 6x + \cos^2 6x = 1 \Leftrightarrow \sin 12x = 0 \ .$$

This equation has as a solution $x = k\pi/12$ for k an integer. Checking each of these for validity, we find that the solutions are $x = \pi/12$, $2\pi/12$, $5\pi/12$, $9\pi/12$, $10\pi/12$, and the general solution is obtained by adding a multiple of π to each of these.

Solution 2. The given equation is equivalent to

$$2\sin 3x\cos 2x - \sin 3x = 2\cos 3x\cos 2x - \cos 3x$$

$$\Leftrightarrow (2\cos 2x - 1)(\sin 3x - \cos 3x) = 0 \ .$$

Thus, either $\cos 2x = \frac{1}{2}$ in which case $x = \pm(\pi/6) + k\pi$ for some integer k, or $\cos 2x \neq \frac{1}{2}$. In the latter case, we must have $\cos 3x \neq 0$ (why?), so that $\tan 3x = 1$ and $x = (\pi/12) + (k\pi/3)$. Thus, all solutions of the equation are $x = (\pi/12) + k\pi$, $(\pi/6) + k\pi$, $(5\pi/12) + k\pi$, $(3\pi/4) + k\pi$ and $(5\pi/6) + k\pi$ where k is an integer.