PROBLEMS FOR MARCH

Please send your solutions to
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no later than **April 15, 2002**.

133. Prove that, if a, b, c, d are real numbers, $b \neq c$, both sides of the equation are defined, and

$$\frac{ac - b^2}{a - 2b + c} = \frac{bd - c^2}{b - 2c + d} ,$$

then each side of the equation is equal to

$$\frac{ad - bc}{a - b - c + d} .$$

Give two essentially different examples of quadruples (a, b, c, d), not in geometric progression, for which the conditions are satisfied. What happens when $b = c$?

134. Suppose that

$$a = zb + yc$$

$$b = xc + za$$

$$c = ya + xb .$$

Prove that

$$\frac{a^2}{1 - x^2} = \frac{b^2}{1 - y^2} = \frac{c^2}{1 - z^2} .$$

Of course, if any of x^2, y^2, z^2 is equal to 1, then the conclusion involves undefined quantities. Give the proper conclusion in this situation. Provide two essentially different numerical examples.

135. For the positive integer n, let $p(n) = k$ if n is divisible by 2^k but not by 2^{k+1}. Let $x_0 = 0$ and define x_n for $n \geq 1$ recursively by

$$\frac{1}{x_n} = 1 + 2p(n) - x_{n-1} .$$

Prove that every nonnegative rational number occurs exactly once in the sequence $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$.

136. Prove that, if in a semicircle of radius 1, five points A, B, C, D, E are taken in consecutive order, then

$$|AB|^2 + |BC|^2 + |CD|^2 + |DE|^2 + |AB||BC||CD| + |BC||CD||DE| < 4 .$$

137. Can an arbitrary convex quadrilateral be decomposed by a polygonal line into two parts, each of whose diameters is less than the diameter of the given quadrilateral?

138. (a) A room contains ten people. Among any three. there are two (mutual) acquaintances. Prove that there are four people, any two of whom are acquainted.

(b) Does the assertion hold if "ten" is replaced by "nine"?