Solutions and Comments

25. Let a, b, c be non-negative numbers such that $a+b+c=1$. Prove that

$$\frac{ab}{c+1}+\frac{bc}{a+1}+\frac{ca}{b+1}\le\frac{1}{4}\ .$$

When does equality hold?

Solution. It is straightforward to show that the inequality holds when one of the numbers is equal to zero. Equality holds if and only if the other two numbers are each equal to $1/2$. Henceforth, assume that all values are positive.

Since $a+b+c=1$, at least one of the numbers is less than $4/9$. Assume that $c<4/9$. Let E denote the left side of the inequality. Then

$$E=abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{a+1}-\frac{1}{b+1}-\frac{1}{c+1}\right)\ .$$

Since

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{1}{4}[(a+1)+(b+1)+(c+1)]\left[\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right]\ge\frac{9}{4}\ ,$$

(by the Cauchy-Schwarz Inequality for example), we have that

$$E\le abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{9}{4}\right)=ab+bc+ca-\frac{9}{4}abc\ .$$

On the other hand, $(1-c)^2=(a+b)^2\ge 4ab$, whence $ab\le\frac{1}{4}(1-c)^2$. Therefore

$$\begin{aligned}E-\frac{1}{4}&\le ab+c(a+b)-\frac{9}{4}abc-\frac{1}{4}\\&=ab\left(1-\frac{9}{4}c\right)+c(1-c)-\frac{1}{4}\\&\le\frac{1}{4}(1-c)^2\left(1-\frac{9}{4}c\right)+c(1-c)-\frac{1}{4}\\&=-\frac{1}{16}c(3c-1)^2\le 0\ .\end{aligned}$$

Equality occurs everywhere if and only if $a=b=c=1/3$.

26. Each of m cards is labelled by one of the numbers $1,2,\cdots,m$. Prove that, if the sum of labels of any subset of cards is not a multiple of $m+1$, then each card is labelled by the same number.

Solution. Let a_k be the label of the kth card, and let $s_n=\sum_{k=1}^{n}a_k$ for $n=1,2,\cdots,m$. Since the sum of the labels of any subset of cards is not a multiple of $m+1$, we get different remainders when we divide the s_n by $m+1$. These remainders must be $1,2,\cdots,m$ in some order. Hence there is an index $i\in\{1,2,\cdots,m\}$ for which $a_2\equiv s_i \pmod{m+1}$. If i were to exceed 1, then we would have a contradiction, since then s_i-a_2 would be a multiple of $m+1$. Therefore, $a_2\equiv s_1=a_1$, so that $a_2\equiv a_1 \pmod{m+1}$, whence $a_2=a_1$. By cyclic rotation of the a_k, we can argue that all of the a_k are equal.

27. Find the least number of the form $|36^m-5^n|$ where m and n are positive integers.

Solution. Since the last digit of 36^m is 6 and the last digit of 5^n is 5, then the last digit of 36^m-5^n is 1 when $36^m>5^n$ and the last digit of 5^n-36^m is 9 when $5^n>36^m$. If $36^m-5^n=1$, then

$$5^n=36^m-1=(6^m+1)(6^m-1)$$

whence 6^m+1 must be a power of 5, an impossibility. 36^m-5^n can be neither -1 nor 9. If $5^n-36^m=9$, then $5^n=9(4\cdot 36^{m-1}+1)$, which is impossible. For $m=1$ and $n=2$, we have that $36^m-5^n=36-25=11$, and this is the least number of the given form.

28. Let A be a finite set of real numbers which contains at least two elements and let $f: A \longrightarrow A$ be a function such that $|f(x)-f(y)|<|x-y|$ for every $x,y\in A$, $x\neq y$. Prove that there is $a\in A$ for which $f(a)=a$. Does the result remain valid if A is not a finite set?

Solution 1. Let $a\in A$, $a_1=f(a)$, and, for $n\geq 2$, $a_n=f(a_{n-1})$. Consider the sequence $\{x_n\}$ with

$$x_n=|a_{n+1}-a_n|$$

where $n=1,2,\cdots$. Since A is a finite set and each a_n belongs to A, there are only a finite number of distinct x_n. Let $x_k=\min_{n\geq 1}\{x_n\}$; we prove by contradiction that $x_k=0$.

Suppose if possible that $x_k>0$. Then

$$x_k=|a_{k+1}-a_k|>|f(a_{k+1})-f(a_k)|=|a_{k+2}-a_{k+1}|=x_{k+1}\ .$$

But this does not agree with the selection of x_k. Hence, $x_k=0$, and this is equivalent to $a_{k+1}=a_k$ or $f(a_k)=a_k$. The desired result follows.

Solution 2. We first prove that $f(A)\neq A$. Suppose, if possible, that $f(A)=A$. Let M be the largest and m be the smallest number in A. Since $f(A)=A$, there are elements a_1 and a_2 in A for which $M=f(a_1)$ and $m=f(a_2)$. Hence

$$M-m=|f(a_1)-f(a_2)|<|a_1-a_2|\leq|M-m|=M-m$$

which is a contradiction. Therefore, $f(A)\subset A$.

Note that $A\supseteq f(A)\supseteq f^2(A)\supseteq\cdots\supseteq f^n(A)\supseteq\cdots$. In fact, we can extend the foregoing argument to show *strict* inclusion as long as the sets in question have more than one element:

$$A\supset f(A)\supset f^2(A)\supset\cdots\supset f^n(A)\supset\cdots\ .$$

(The superscripts indicate multiple composites of f.) Since A is a finite set, there must be a positive integer m for which $f^m(A)=\{a\}$, so that $f^{m+1}(A)=f^m(A)$. Thus, $f(a)=a$.

Example. If A is not finite, the result may fail. Indeed, we can take $A=(0,\frac{1}{2})$ (the open interval of real numbers strictly between 0 and $\frac{1}{2}$) or $A=\{2^{-2n}:n=1,2,\cdots\}$ and $f(x)=x^2$.

29. Let A be a nonempty set of positive integers such that if $a\in A$, then $4a$ and $\lfloor\sqrt{a}\rfloor$ both belong to A. Prove that A is the set of all positive integers.

Solution. (i) Let us first prove that $1\in A$. Let $a\in A$. Then we have

$$\lfloor a^{1/2}\rfloor\in A\ ,\quad \lfloor\lfloor a^{1/2}\rfloor^{1/2}\rfloor\in A\ ,\quad\cdots\quad ,\lfloor\cdots\lfloor a^{1/2}\rfloor^{1/2}\rfloor^{1/2}\cdots\rfloor\in A\ ,\quad\cdots\ .$$

Also, the following inequalities are true

$$1\leq\lfloor a^{1/2}\rfloor\leq a^{1/2}\ ,\quad 1\leq\lfloor\lfloor a^{1/2}\rfloor^{1/2}\rfloor\leq a^{1/2^2}\ ,\quad\cdots\ ,\quad 1\leq\lfloor\cdots\lfloor a^{1/2}\rfloor^{1/2}\rfloor^{1/2}\cdots\rfloor\leq a^{1/2^n}\ ,$$

where there are n brackets in the general inequality. There is a sufficiently large positive integer k for which $a^{1/2^k}\leq 1.5$, and for this k, we have, with k brackets,

$$1\leq\lfloor\cdots\lfloor a^{1/2}\rfloor^{1/2}\rfloor^{1/2}\cdots\rfloor\leq a^{1/2^k}\leq 1.5\ ,$$

and thus

$$\lfloor \cdots \lfloor \lfloor a^{1/2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor = 1 \ ,$$

(ii) We next prove that $2^n \in A$ for $n = 1, 2, \cdots$. Indeed, since $1 \in A$, we obtain that, for each positive integer n, $2^{2n} \in A$ so that $2^n = \lfloor \sqrt{2^{2n}} \rfloor \in A$.

(iii) We finally prove that an arbitrary positive integer m is in A. It suffices to show that there is a positive integer k for which $m^{2^k} \in A$. For each positive integer k, there is a positive integer p_k such that $2^{p_k} \leq m^{2^k} < 2^{p_k+1}$ (we can take $p_k = \lfloor \log_2 m^{2^k} \rfloor$). For k sufficiently large, we have the inequality

$$\left(1 + \frac{1}{m}\right)^{2^k} \geq 1 + \frac{1}{m} \cdot 2^k > 4 \ .$$

This combined with the foregoing inequality produces

$$2^{p_k} \leq m^{2^k} < 2^{p_k+1} < 2^{p_k+2} < (m+1)^{2^k} \ . \tag{$*$}$$

Since $2^{2(p_k+1)+1} \in A$, we have that

$$\lfloor \sqrt{2^{2(p_k+1)+1}} \rfloor = \lfloor 2^{p_k+1}\sqrt{2} \rfloor \in A \ .$$

Hence, with $k+1$ brackets,

$$\lfloor \cdots \lfloor \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor \in A \ .$$

On the other hand, using $(*)$, we get

$$m^{2^k} < 2^{p_k+1} \leq \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor < (m+1)^{2^k}$$

and, then, with $k+1$ brackets,

$$m \leq \lfloor \cdots \lfloor \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor < m+1 \ .$$

Thus,

$$m = \lfloor \cdots \lfloor \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor \in A \ .$$

30. Find a point M within a regular pentagon for which the sum of its distances to the vertices is minimum.

Solution. We solve this problem for the regular $n-$gon $A_1 A_2 \cdots A_n$. Choose a system of coordinates centred at O (the circumcentre) such that

$$A_k \sim \left(r\cos\frac{2k\pi}{n}, r\sin\frac{2k\pi}{n} \right) \ , \quad r = \|\overrightarrow{OA_k}\|$$

for $k = 1, 2, \cdots, n$. Then

$$\sum_{k=1}^{n} \overrightarrow{OA_k} = \sum_{k=1}^{n} \left(r\cos\frac{2k\pi}{n}, r\sin\frac{2k\pi}{n} \right) = \left(r\sum_{k=1}^{n}\cos\frac{2k\pi}{n}, r\sum_{k=1}^{n}\sin\frac{2k\pi}{n} \right) = (0,0) \ .$$

Indeed, letting $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and using DeMoivre"s Theorem, we have that $\zeta^n = 1$ and

$$\begin{aligned}\sum_{k=1}^{n}\cos\frac{2k\pi}{n} + i\sum_{k=1}^{n}\sin\frac{2k\pi}{n} &= \sum_{k=1}^{n}\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k \\ &= \sum_{k=1}^{n}\zeta^k = \zeta\left(\frac{1-\zeta^n}{1-\zeta}\right) = 0 \ ,\end{aligned}$$

whence $\sum_{k=1}^{n} \cos(2\pi k/n) = \sum_{k=1}^{n} \sin(2\pi k/n) = 0$. On the other hand,

$$\begin{aligned}
\sum_{k=1}^{n} \|\overrightarrow{MA_k}\| &= \frac{1}{r}\sum_{k=1}^{n} \|\overrightarrow{OA_k} - \overrightarrow{OM}\|\|\overrightarrow{OA_k}\| \\
&\geq \frac{1}{r}\sum_{k=1}^{n} (\overrightarrow{OA_k} - \overrightarrow{OM}) \cdot \overrightarrow{OA_k} \\
&= \frac{1}{r}\left(\sum_{k=1}^{n} \|\overrightarrow{OA_k}\|^2 - \|\overrightarrow{OM}\| \sum_{k=1}^{n} \overrightarrow{OA_k}\right) \\
&= \sum_{k=1}^{n} r = \sum_{k=1}^{n} \|\overrightarrow{OA_k}\| \ .
\end{aligned}$$

Equality occurs if and only if $M = O$, so that O is the desired point.