PROBLEMS FOR MAY

Solutions should be submitted to
Dr. Valeria Pandelieva
708 - 195 Clearview Avenue
Ottawa, ON K1Z 6S1
Solution to these problems should be postmarked no later than **June 30, 2000**.

Notes: A set of lines of *concurrent* if and only if they have a common point of intersection.

7. Let
$$S = \frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \frac{3^2}{5 \cdot 7} + \cdots + \frac{500^2}{999 \cdot 1001} \ .$$
Find the value of S.

8. The sequences $\{a_n\}$ and $\{b_n\}$ are such that, for every positive integer n,
$$a_n > 0 \ , \quad b_n > 0 \ , \quad a_{n+1} = a_n + \frac{1}{b_n} \ , \quad b_{n+1} = b_n + \frac{1}{a_n} \ .$$
Prove that $a_{50} + b_{50} > 20$.

9. There are six points in the plane. Any three of them are vertices of a triangle whose sides are of different length. Prove that there exists a triangle whose smallest side is the largest side of another triangle.

10. In a rectangle, whose sides are 20 and 25 units of length, are placed 120 squares of side 1 unit of length. Prove that a circle of diameter 1 unit can be placed in the rectangle, so that it has no common points with the squares.

11. Each of nine lines divides a square into two quadrilaterals, such that the ratio of their area is 2:3. Prove that at least three of these lines are concurrent.

12. Each vertex of a regular 100-sided polygon is marked with a number chosen from among the natural numbers $1, 2, 3, \cdots, 49$. Prove that there are four vertices (which we can denote as A, B, C, D with respective numbers a, b, c, d) such that $ABCD$ is a rectangle, the points A and B are two adjacent vertices of the rectangle and $a + b = c + d$.

PROBLEMS FOR APRIL

Solutions should be submitted to Prof. E.J. Barbeau, Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3. Electronic solutions should be submitted either in plain text or in TeX; please do not use other word-processing software; the email address is `barbeau@math.utoronto.ca`. Solutions should be delivered by hand or mail with a postmark no later than **May 31, 2000**.

Notes: The *inradius* of a triangle is the radius of the *incircle*, the circle that touches each side of the polygon. The *circumradius* of a triangle is the radius of the *circumcircle*, the circle that passes through its three vertices.

1. Let M be a set of eleven points consisting of the four vertices along with seven interior points of a square of unit area.

   (a) Prove that there are three of these points that are vertices of a triangle whose area is at most $1/16$.

   (b) Give an example of a set M for which no four of the interior points are collinear and each nondegenerate triangle formed by three of them has area at least $1/16$.

2. Let a, b, c be the lengths of the sides of a triangle. Suppose that $u = a^2 + b^2 + c^2$ and $v = (a + b + c)^2$. Prove that

$$\frac{1}{3} \leq \frac{u}{v} < \frac{1}{2}$$

and that the fraction 1/2 on the right cannot be replaced by a smaller number.

3. Suppose that $f(x)$ is a function satisfying

$$|f(m + n) - f(m)| \leq \frac{n}{m}$$

for all rational numbers n and m. Show that, for all natural numbers k,

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \leq \frac{k(k-1)}{2} .$$

4. Is it true that any pair of triangles sharing a common angle, inradius and circumradius must be congruent?

5. Each point of the plane is coloured with one of 2000 different colours. Prove that there exists a rectangle all of whose vertices have the same colour.

6. Let n be a positive integer, P be a set of n primes and M a set of at least $n + 1$ natural numbers, each of which is divisible by no primes other than those belonging to P. Prove that there is a nonvoid subset of M, the product of whose elements is a square integer.