2019 CMO Qualifying Repêchage

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Official Problem Set

1. A function f is called injective if when $f(n) = f(m)$, then $n = m$. Suppose that f is injective and
$$\frac{1}{f(n)} + \frac{1}{f(m)} = \frac{4}{f(n) + f(m)}.$$
Prove $m = n$.

2. Rosemonde is stacking spheres to make pyramids. She constructs two types of pyramids S_n and T_n. The pyramid S_n has n layers, where the top layer is a single sphere and the i^{th} layer is an $i \times i$ square grid of spheres for each $2 \leq i \leq n$. Similarly, the pyramid T_n has n layers where the top layer is a single sphere and the i^{th} layer is $\frac{i(i+1)}{2}$ spheres arranged into an equilateral triangle for each $2 \leq i \leq n$.

   If all the spheres have radius 2, determine the smallest n so that the difference between the height of S_n and the height of T_n is greater than 2019.

3. Let $f(x) = x^3 + 3x^2 - 1$ have roots a, b, c.

   (a) Find the value of $a^3 + b^3 + c^3$.

   (b) Find all possible values of $a^2b + b^2c + c^2a$.

4. Let n be a positive integer. For a positive integer m, we partition the set $\{1, 2, 3, \ldots, m\}$ into n subsets, so that the product of two different elements in the same subset is never a perfect square. In terms of n, find the largest positive integer m for which such a partition exists.

5. Let (m, n, N) be a triple of positive integers. Bruce and Duncan play a game on an $m \times n$ array, where the entries are all initially zeroes. The game has the following rules.

   - The players alternate turns, with Bruce going first.
   - On Bruce's turn, he picks a row and either adds 1 to all of the entries in the row or subtracts 1 from all the entries in the row.
   - On Duncan's turn, he picks a column and either adds 1 to all of the entries in the column or subtracts 1 from all of the entries in the column.
   - Bruce wins if at some point there is an entry x with $|x| \geq N$.

   Find all triples (m, n, N) such that no matter how Duncan plays, Bruce has a winning strategy.

6. Pentagon $ABCDE$ is given in the plane. Let the perpendicular from A to line CD be F, the perpendicular from B to DE be G, from C to EA be H, from D to AB be I, and from E to BC be J. Given that lines AF, BG, CH, and DI concur, show that they also concur with line EJ.

7. There are n passengers in a line, waiting to board a plane with n seats. For $1 \leq k \leq n$, the k^{th} passenger in line has a ticket for the k^{th} seat. However, the first passenger ignores his ticket, and decides to sit in a seat at random. Thereafter, each passenger sits as follows: If his/her assigned is empty, then he/she sits in it. Otherwise, he/she sits in an empty seat at random. How many different ways can all n passengers be seated?

8. For $t \geq 2$, define $S(t)$ as the number of times t divides into $t!$. We say that a positive integer t is a *peak* if $S(t) > S(u)$ for all values of $u < t$.

   Prove or disprove the following statement:

   For every prime p, there is an integer k for which p divides k and k is a peak.