2017 CMO Qualifying Repêchage

Sun Life Financial

Official Solutions

1. Malcolm writes a positive integer on a piece of paper. Malcolm doubles this integer and subtracts 1, writing this second result on the same piece of paper. Malcolm then doubles the second integer and adds 1, writing this third integer on the paper. If all of the numbers Malcolm writes down are prime, determine all possible values for the first integer.

   Solution: Let n be the first integer. Then $2n-1$ is the second and $4n-1$ is the third. We can write n as $3k$, $3k+1$, or $3k+2$ for some non-negative integer k.

   When $n = 3k$, we must have $k = 1$, since n is a prime. In this instance the three number are 3, 5, 11, which are all prime.

   When $n = 3k+1$, we have $4n-1 = 12k+3 = 3(4k+3)$. Since this is a multiple of 3, it must be 3, and we have $k = 0$. This means $n = 1$, and the first number written down is not prime.

   When $n = 3k+2$, we have $2n-1 = 6k+3 = 3(2k+1)$. Since this is a multiple of 3, it must be 3, and we have $k = 0$. In this instance, the three numbers are 2, 3, 7, which are all prime.

   Thus, the first integer is either 2 or 3.

2. For any positive integer n, let $\phi(n)$ be the number of integers in the set $\{1, 2, \ldots, n\}$ whose greatest common divisor with n is 1. Determine the maximum value of $\frac{n}{\phi(n)}$ for n in the set $\{2, \ldots, 1000\}$ and all values of n for which this maximum is attained.

   Solution: Suppose the prime divisors of n are $p_1, p_2, \ldots, p_k$. Then

   $$\frac{n}{\phi(n)} = \left(\frac{p_1}{p_1-1}\right) \cdot \left(\frac{p_2}{p_2-1}\right) \cdots \left(\frac{p_k}{p_k-1}\right)$$

   Since the function $f(x) = \dfrac{x}{x-1}$ is positive and decreasing for $x > 0$, we must choose our integer so that it has as many small positive prime divisors as possible. Since $n \le 1000$ we can have at most 4 prime factors, with 2, 3, 5, and 7 being the smallest. This gives

   $$\frac{n}{\phi(n)} = 2 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = \frac{35}{8}.$$

   The numbers less than 1000 which have 2, 3, 5, and 7 as factors are the multiples of 210. So the maximum is attained for 210, 420, 630, and 840.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following equation for all $x, y \in \mathbb{R}$.

   $$(x+y)f(x-y) = f(x^2 - y^2).$$

   Solution: Let t be a real number, and let $x = \frac{t+1}{2}$ and $y = \frac{t-1}{2}$. Substituting into the given equation yields $f(t) = tf(1)$ for every real t. If we let $f(1) = m$ then this gives that $f(t) = mt$ for any real number m.

4. In this question we re-define the operations addition and multiplication as follows: $a + b$ is defined as the minimum of a and b, while $a * b$ is defined to be the sum of a and b. For example, $3+4 = 3$, $3*4 = 7$, and $3*4^2+5*4+7 = \min(3 \text{ plus } 4 \text{ plus } 4, 5 \text{ plus } 4, 7) = \min(11, 9, 7) = 7$.

   Let a, b, c be real numbers. Characterize, in terms of a, b, c, what the graph of $y = ax^2+bx+c$ looks like.

   Solution: $y = ax^2 + bx + c$ becomes $y = \min(2x + a, x + b, c)$.

   When $2b = a + c$, the three lines $y = 2x + a, y = x + b$, and $y = c$ intersect at the point $(c - b, c)$. Moreover, if $2b > a + c$, the line $y = x + b$ is always above at least one of the other two lines. When that occurs, the graph is:

$$\begin{array}{ll} 2x + a & x < \frac{c-a}{2} \\ c & x \geq \frac{c-a}{2} \end{array}$$

   When $2b < a + b$, the line $y = x + b$ is beneath the other two lines when $b - a < x < c - b$. In this instance, the graph is:

$$\begin{array}{ll} 2x + a & x \leq b - a \\ x + b & b - a < x < c - b \\ c & c - b \leq x \end{array}$$

5. Prove that for all real numbers x, y,

$$(x^2 + 1)(y^2 + 1) + 4(x - 1)(y - 1) \geq 0.$$

   Determine when equality holds.

   Solution: Expanding and refactoring the left hand side of the given inequality gives

$$(xy + 1)^2 + (x + y - 2)^2.$$

   This is clearly non-negative, so the inequality holds. This holds with equality when each of the terms is 0. So $xy = -1$ and $x + y = 2$. Solving this system yields $(x, y) = (1 - \sqrt{2}, 1 + \sqrt{2}), (1 + \sqrt{2}, 1 - \sqrt{2})$.

6. Let N be a positive integer. There are N tasks, numbered $1, 2, 3, \ldots, N$, to be completed. Each task takes one minute to complete and the tasks must be completed subjected to the following conditions:

   • Any number of tasks can be performed at the same time.

   • For any positive integer k, task k begins immediately after all tasks whose numbers are divisors of k, not including k itself, are completed.

   • Task 1 is the first task to begin, and it begins by itself.

   Suppose $N = 2017$. How many minutes does it take for all of the tasks to complete? Which tasks are the last ones to complete?

Solution: Given any positive integer $n > 1$, let $f(n)$ be the number of prime factors (not necessarily distinct) in the prime factorization of n. We claim that task n must wait $f(n)$ minutes before beginning. We will prove this by induction on the number of prime factors of n, i.e. $f(n)$. All numbers n that satisfy $f(n) = 1$ are prime numbers. Since the only divisors of a prime number are 1 and itself, these tasks only have to wait for task 1 to complete before beginning. Since task 1 takes one minute, task n only need to wait one minute. Hence, the claim holds for all numbers with 1 prime factor. Now suppose this holds for all tasks with at most k prime factors. Let n be a number with $k+1$ prime factors. All divisors of n (not equal to n), have at most k prime factors. By the induction hypothesis, the tasks for these divisors are either completed, or waited k minutes to begin. Since each task is one minute long, all of these tasks will be completed after $k+1$ minutes. This completes the induction proof to the claim.

For $N = 2017$, note that $2^{10} = 1024$ is the largest power of 2 smaller than 2017. This implies that all numbers ≤ 2017 have at most 10 prime factors, since the smallest number with 11 prime factors is $2^{11} = 2048 > 2017$. Hence, by the claim, all tasks have to wait at most 10 minutes to begin, with task 1024 having to wait for these 10 minutes to begin. Hence, the tasks take 11 minutes to complete.

Finally, the final tasks to complete are those that have 10 prime factors. The smallest such task is $2^{10} = 1024$. The next biggest number with 10 prime factors is $2^9 \times 3 = 1536$. The next candidates are $2^8 \times 3^2 = 256 \times 9 = 2304$ and $2^9 \times 5 = 2560$, which are all larger than 2017. Hence, tasks 1024 and 1536 are the final tasks to complete.

7. Given a set $S_n = \{1, 2, 3, \ldots, n\}$, we define a *preference list* to be an ordered subset of S_n. Let P_n be the number of preference lists of S_n. Show that for positive integers $n > m$, $P_n - P_m$ is divisible by $n - m$.

   Note: the empty set and S_n are subsets of S_n.

   Solution: Note that if $n = m + 1$ the result is clearly true, so assume $n - m > 1$. Let T_n be the set of preference lists of S_n, and T_n, m to be the set of preference lists of S_n which contain an element of S_n larger than m. Observe that $T_n = T_m \cup T_{n,m}$ and so $P_n - P_m = |T_{n,m}|$.

   Let $T_{n,m}^k$ be the preference lists of $T_{n,m}$ for which the first element in the list larger than m is k. Consider a bijection on S_n which maps k to j, j to k, and each other element to itself. Then this will map $T_{n,m}^k$ to Tn, m^j and vice-versa. The mapping of elements between the sets is clearly one-to-one in each direction, and so is a bijection. Thus $|T_{n,m}^k| = |T_{n,m}^j|$ for any $j, k \in \{m+1, m+2, \ldots, n\}$. Since $T_{n,m} = \bigcup_{k=m+1}^{n} T_{n,m}^k$ and each $T_{n,m}^k$ has the same size, $|T_{n,m}|$ is divisible by $n - m$.

8. A convex quadrilateral $ABCD$ is said to be *dividable* if for every internal point P, the area of ΔPAB plus the area of ΔPCD is equal to the area of ΔPBC plus the area of ΔPDA. Characterize all quadrilaterals which are dividable

   Solution: Suppose $ABCD$ is a parallelogram. Then for any point P inside $ABCD$, the sum of the areas of $\triangle PAB$ and $\triangle PCD$ is half the area of $ABCD$ and so equal to the sum of the areas of $\triangle PBC$ and $\triangle PDA$.

   Suppose $ABCD$ is dividable, and consider a line ℓ parallel to AB which intersects the interior of $ABCD$. For any two points P and Q on ℓ, the area of $\triangle PAB$ is the same as the area of $\triangle QAB$. In order for this to be dividable, $\triangle PCD$ and $\triangle QCD$ must also have the same areas. This only happens if ℓ is also parallel to CD. Thus, AB is parallel to CD. Similarly, we find that BC is parallel to DA and so $ABCD$ is a parallelogram. Therefore, $ABCD$ is dividable if and only if it is a parallelogram.