

1. (a) Find all positive integers n such that $11|(3^n + 4^n)$.

Solution: We want to find all n such that $3^n \equiv -4^n \pmod{11}$. Multiplying both sides of this equation by 3^n we get $9^n \equiv -1 \pmod{11}$. However $9^5 \equiv 1 \pmod{11}$, and there is no smaller n such that $9^n \equiv -1 \pmod{11}$. Thus, there are no such positive integers n .

- (b) Find all positive integers n such that $31|(4^n + 7^n + 20^n)$.

Solution: Since 7, 31 are coprime, $0 \equiv 4^n + 7^n + 20^n \pmod{31}$ is equivalent to

$$\begin{aligned} 0 &\equiv \frac{4^n + 7^n + 20^n}{7^n} && \pmod{31} \\ &\equiv \left(\frac{20}{35}\right)^n + 1 + \left(\frac{100}{35}\right)^n && \pmod{31} \\ &\equiv 1 + \left(\frac{20}{4}\right)^n + \left(\frac{100}{4}\right)^n && \pmod{31} \\ &\equiv 1 + 5^n + 25^n && \pmod{31} \\ &\equiv 1 + 5^n + 5^{2n} && \pmod{31} \end{aligned}$$

Since $5^3 \equiv 1 \pmod{31}$, this sum only depends on $n \pmod{3}$. We calculate:

$$\begin{aligned} 1 + 5^0 + 5^{2*0} &\equiv 3 && \pmod{31} \\ 1 + 5^1 + 5^{2*1} &\equiv 0 && \pmod{31} \\ 1 + 5^2 + 5^{2*2} &\equiv 0 && \pmod{31} \end{aligned}$$

Hence the answer is all n such that n is not a multiple of 3.

2. Let $P = (7, 1)$ and let $O = (0, 0)$.

- (a) If S is a point on the line $y = x$ and T is a point on the horizontal x-axis so that P is on the line segment ST , determine the minimum possible area of triangle OST .

Solution: Let (a, a) be the coordinates of S and $(b, 0)$ the coordinates of T . The line segments SP , PT , and ST all have the same slope, so $\frac{a-1}{a-7} = \frac{1}{7-b}$. Solving for b we get $b = \frac{6a}{a-1}$.

The triangle has a base of $OT = b$ and a height of a , so the area is $A = \frac{6a^2}{2(a-1)} = \frac{3a^2}{a-1} = 3a + 3 + \frac{3}{a-1} = 3(a-1) + 6 + \frac{3}{a-1}$. By the AM-GM inequality, $A \geq 2\sqrt{3(a-1)\frac{3}{a-1}} + 6 = 12$, with equality when $3(a-1) = \frac{3}{a-1}$. Equality holds when $a = 2$, and we have $a = 2, b = 12$ and an area of 12.

- (b) If U is a point on the line $y = x$ and V is a point on the horizontal x -axis so that P is on the line segment UV , determine the minimum possible perimeter of triangle OUV .

Solution: Consider a circle that is tangent to the lines OU , OV , and UV at the points A , B , and C respectively. We notice that $AU = CU$ and $BV = CV$, so the perimeter of OUV equals $OA + OB = 2OA$.

In order to minimize this sum, we seek to minimize the size of the circle. Notice that if the circle is tangent to UV at a point that is not P , then there exists a line through P that does not touch the circle. Taking this line as UV would allow for the circle to be smaller. Thus, the perimeter will be minimized when the circle is tangent to the point P .

Let $U = (a, a)$, $V = (b, 0)$, and $OA = x$. Then $VB = VC$, so $\sqrt{(b-7)^2 + 1} = x - b$. Solving for b yields $b = \frac{x^2 - 50}{2x - 14}$. Similarly $UP = UA$ and we find that $a = \frac{x^2 - 50}{2\sqrt{2}x - 16}$.

As in part (a), we have $b = \frac{6a}{a-1}$. Substituting in a and b to this equation gives:

$$\begin{aligned} \frac{x^2 - 50}{2x - 14} &= \frac{6 \frac{x^2 - 50}{2\sqrt{2}x - 16}}{\frac{x^2 - 50}{2\sqrt{2}x - 16} - 1} \\ \frac{x^2 - 50}{2x - 14} &= \frac{6 \frac{x^2 - 50}{2\sqrt{2}x - 16}}{\frac{x^2 - 2\sqrt{2}x - 34}{2\sqrt{2}x - 16}} \\ \frac{x^2 - 50}{2x - 14} &= \frac{6(x^2 - 50)}{x^2 - 2\sqrt{2}x - 34} \\ (x^2 - 50)(x^2 - 2\sqrt{2}x - 34) &= (x^2 - 50)(12x - 84) \\ (x^2 - 50)(x^2 - (2\sqrt{2} + 12)x + 50) &= 0 \end{aligned}$$

Solving this for x yields: $x = \pm\sqrt{50}$, $x = 6 + \sqrt{2} \pm 2\sqrt{3(\sqrt{2} - 1)}$

When $x = \pm\sqrt{50}$, a and b are both 0, which is not a solution. Of the two remaining solutions, we take the larger one, as the other occurs from the circle that is contained inside triangle OUV . thus the minimal perimeter is $2(6 + \sqrt{2} + 2\sqrt{3(\sqrt{2} - 1)})$.

3. Given an $n \times n \times n$ grid of unit cubes, a cube is *good* if it is a sub-cube of the grid and has side length at least two. If a good cube contains another good cube and their faces do not intersect, the first good cube is said to *properly* contain the second. What is the size of the largest possible set of good cubes such that no cube in the set properly contains another cube in the set?

Solution: Let S be the set of good cubes with side length 2 or 3. For any $s \in S$, we define C_s to be the set of good cubes that have s as their centre. Then we can see that for any s and any pair of distinct cubes in C_s , that one cube properly contains the other. Also, for any good cube c_1 in the grid, there exists a cube $c_2 \in S$ such that c_2 is the centre of c_1 . Thus, the size of largest possible set of good cubes with no cube properly containing each other is at most $|S|$. Finally, we observe that no cube in S can properly contain another cube in S . Thus $|S| = (n - 1)^3 + (n - 2)^3$ is the size of the largest possible set.

4. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + f(y)) + f(x - f(y)) = x.$$

Solution: Let $a = f(0)$ and let $y = 0, x = a$. The given equation becomes $f(2a) + f(0) = a$ which is equivalent to $f(2a) = 0$. Now, let $y = 2a$ and we get $2f(x) = x$ or $f(x) = x/2$. If we substitute this into the given relation, we see that the equation holds with equality. Thus, $f(x) = x/2$ is the only such function.

5. Consider a convex polygon P with n sides and perimeter P_0 . Let the polygon Q , whose vertices are the midpoints of the sides of P , have perimeter P_1 . Prove that $P_1 \geq \frac{P_0}{2}$.

Solution: When P_0 is a triangle, the sides of P_1 are parallel to the sides of P_0 and exactly half of the length of the side they are parallel to, so the result holds. When P_0 has at least 4 sides, label the vertices v_0, v_1, \dots, v_{n-1} clockwise. Label the midpoints m_0, m_1, \dots, m_{n-1} clockwise, where m_0 is the midpoint of v_0v_1 . We observe that $2m_im_{i+1} = v_iv_{i+2}$, so it is equivalent to show that $v_0v_2 + v_1v_3 + \dots + v_{n-1}v_1 \geq v_0v_1 + v_1v_2 + \dots + v_{n-1}v_0$.

The segment v_iv_{i+2} intersects v_iv_{i+1} at a point w and intersects $v_{i+1}v_{i+2}$ at the point x . Observe that $v_iw + wv_{i+1} \geq v_iv_{i+1}$ by the triangle inequality. We can continue this all around the the polygon. We observe that over all segments v_iv_{i+1} , the segments v_iw and wv_{i+1} are disjoint subsegments of the set of segments v_iv_{i+2} . Thus, we have $v_0v_2 + v_1v_3 + \dots + v_{n-1}v_1 \geq v_0v_1 + v_1v_2 + \dots + v_{n-1}v_0$ and $P_1 \geq \frac{P_0}{2}$.

6. Determine all ordered triples of positive integers (x, y, z) such that $\gcd(x + y, y + z, z + x) > \gcd(x, y, z)$.

Solution: Let $g = \gcd(x+y, y+z, z+x)$. We observe that g must divide $(x+y) + (y+z) - (x+z) = 2y$. Similarly, g must also divide $2x$ and $2z$. Thus g divides $2 \gcd(x, y, z)$. In order to have $g > \gcd(x, y, z)$ we must have $g = 2 \gcd(x, y, z)$.

Assume that $\gcd(x, y, z) = 1$. Then 2 must divide $x + y, y + z$, and $z + x$. This can only happen when all of x, y, z have the same parity. Since their gcd is 1, they must therefore all be odd. Thus, we have that x, y, z are odd numbers multiplied by a common factor. This is equivalent to having the largest power of 2 that divides each number being the same.

7. Starting at $(0, 0)$, Richard takes $2n+1$ steps, with each step being one unit either East, North, West, or South. For each step, the direction is chosen uniformly at random from the four possibilities. Determine the probability that Richard ends at $(1, 0)$.

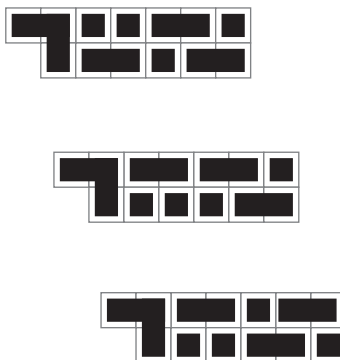
Solution: In order for Richard to end up at $(1, 0)$ he must take the same number of steps north as he does south, and the number of steps he takes east must be one more than the number he takes west. Thus, we see that the number of steps taken north or west is n as is the number taken south or west. The number of ways to choose the n steps that are north or west is $\binom{2n+1}{n}$, which is the

same as the number of ways to choose the steps that are south or west. A step is west if and only if it is chosen in both sets. Since there are a total of $2n + 1$ steps, the probability of ending at $(1, 0)$ will be $\frac{\binom{2n+1}{n}^2}{4^{2n+1}}$.

8. Let $n \geq 3$ be a positive integer. A *chipped n -board* is a $2 \times n$ checkerboard with the bottom left square removed. Lino wants to tile a chipped n -board and is allowed to use the following types of tiles:

- Type 1: Any $1 \times k$ board where $1 \leq k \leq n$
- Type 2: Exactly one chipped k -board where $2 \leq k \leq n$, which covers the left-most tile.

Two tilings T_1 and T_2 are considered the same if there is a set of consecutive Type 1 tiles in both rows of T_1 that can be vertically swapped to obtain the tiling T_2 . For example, the following three tilings of a chipped 7-board are the same:



For any positive integer n and any positive integer $1 \leq m \leq 2n - 1$, let $c_{m,n}$ be the number of distinct tilings of a chipped n -board using exactly m tiles (any combination of tile types may be used), and define the polynomial

$$P_n(x) = \sum_{m=1}^{2n-1} c_{m,n} x^m.$$

Find, with justification, polynomials $f(x)$ and $g(x)$ such that

$$P_n(x) = f(x)P_{n-1}(x) + g(x)P_{n-2}(x)$$

for all $n \geq 3$.

Solution: Given a chipped $(n - 1)$ -board, we can extend this to a chipped n -board in 4 different ways. For each of the two new spots in the last column, we can either have the tile be attached to the previous column, or we can have it be a new tile. This gives a recurrence of $P_n(x) =$

$(1+2x+x^2)P_{n-1}(x)$. However, note that we have overcounted, as some of these tilings are equivalent to each other and some are not valid.

We get equivalent tilings when the k rightmost columns of the $(n-1)$ -board contain two $1 \times k$ boards, in which case the tilings that consist of attaching one from the last column and not attaching the other are equivalent. Note that the number of such overcountings is given by the polynomial x^3P_{n-1-k} , as this tells us how many ways we can tile the left-most $n-k-1$ columns.

Also, note that if the $(n-1)$ -board is a single piece, then neither of the two constructions which attach a single piece is valid. This means that there are two invalid tilings of size 2, which has polynomial $2x^2$. Thus, our recurrence is actually

$$P_n(x) = (1 + 2x + x^2)P_{n-1}(x) - x^3(P_{n-2}(x) + P_{n-3}(x) + \cdots + P_1(x)) - 2x^2.$$

We can write a similar recurrence for $P_{n-1}(x)$:

$$P_{n-1}(x) = (1 + 2x + x^2)P_{n-2}(x) - x^3(P_{n-3}(x) + P_{n-4}(x) + \cdots + P_1(x)) - 2x^2.$$

Subtracting these recurrences gives:

$$P_n(x) - P_{n-1}(x) = (1 + 2x + x^2)(P_{n-1}(x) - P_{n-2}(x)) - x^3P_{n-2}(x).$$

We can simplify this to:

$$P_n(x) = (2 + 2x + x^2)P_{n-1}(x) - (1 + 2x + x^2 + x^3)P_{n-2}(x).$$