1. (a) Find all positive integers n such that $11|(3^n + 4^n)$.

    (b) Find all positive integers n such that $31|(4^n + 7^n + 20^n)$.

2. Let $P = (7, 1)$ and let $O = (0, 0)$.

    (a) If S is a point on the line $y = x$ and T is a point on the horizontal x-axis so that P is on the line segment ST, determine the minimum possible area of triangle OST.

    (b) If U is a point on the line $y = x$ and V is a point on the horizonal x-axis so that P is on the line segment UV, determine the minimum possible perimeter of triangle OUV.

3. Given an $n \times n \times n$ grid of unit cubes, a cube is *good* if it is a sub-cube of the grid and has side length at least two. If a good cube contains another good cube and their faces do not intersect, the first good cube is said to *properly* contain the second. What is the size of the largest possible set of good cubes such that no cube in the set properly contains another cube in the set?

4. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) + f(x - f(y)) = x.$$

5. Consider a convex polygon P with n sides and perimeter P_0. Let the polygon Q, whose vertices are the midpoints of the sides of P, have perimeter P_1. Prove that $P_1 \geq \frac{P_0}{2}$.

6. Determine all ordered triples of positive integers (x, y, z) such that $\gcd(x + y, y + z, z + x) > \gcd(x, y, z)$.

7. Starting at $(0, 0)$, Richard takes $2n+1$ steps, with each step being one unit either East, North, West, or South. For each step, the direction is chosen uniformly at random from the four possibilities. Determine the probability that Richard ends at $(1, 0)$.

8. Let $n \geq 3$ be a positive integer. A *chipped n-board* is a $2 \times n$ checkerboard with the bottom left square removed. Lino wants to tile a chipped n-board and is allowed to use the following types of tiles:

- Type 1: any $1 \times k$ board where $1 \leq k \leq n$

- Type 2: any chipped k-board where $1 \leq k \leq n$ that must cover the left-most tile of the $2 \times n$ checkerboard.

Two tilings T_1 and T_2 are considered the same if there is a set of consecutive Type 1 tiles in both rows of T_1 that can be vertically swapped to obtain the tiling T_2. For example, the following three tilings of a chipped 7-board are the same:
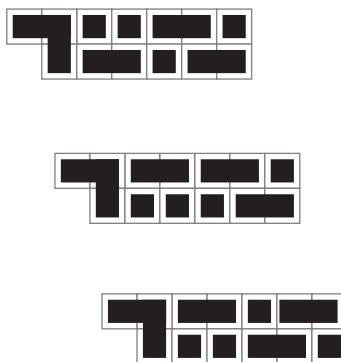
For any positive integer n and any positive integer $1 \leq m \leq 2n - 1$, let $c_{m,n}$ be the number of distinct tilings of a chipped n-board using exactly m tiles (any combination of tile types may be used), and define the polynomial

$$P_n(x) = \sum_{m=1}^{2n-1} c_{m,n} x^m.$$

Find, with justification, polynomials $f(x)$ and $g(x)$ such that

$$P_n(x) = f(x)P_{n-1}(x) + g(x)P_{n-2}(x)$$

for all $n \geq 3$.