

Canadian Mathematical Olympiad Qualification Repechage

1 Determine all real solutions to the following equation:

$$2^{(2^x)} - 3 \cdot 2^{(2^{x-1}+1)} + 8 = 0.$$

2 In triangle ABC , $\angle A = 90^\circ$ and $\angle C = 70^\circ$. F is point on AB such that $\angle ACF = 30^\circ$, and E is a point on CA such that $\angle CFE = 20^\circ$. Prove that BE bisects $\angle B$.

3 A positive integer n has the property that there are three positive integers x, y, z such that $\text{lcm}(x, y) = 180$, $\text{lcm}(x, z) = 900$ and $\text{lcm}(y, z) = n$, where lcm denotes the lowest common multiple. Determine the number of positive integers n with this property.

4 Four boys and four girls each bring one gift to a Christmas gift exchange. On a sheet of paper, each boy randomly writes down the name of one girl, and each girl randomly writes down the name of one boy. At the same time, each person passes their gift to the person whose name is written on their sheet. Determine the probability that *both* of these events occur:

- (i) Each person receives exactly one gift;
- (ii) No two people exchanged presents with each other (i.e., if A gave his gift to B , then B did not give her gift to A).

5 For each positive integer k , let $S(k)$ be the sum of its digits. For example, $S(21) = 3$ and $S(105) = 6$. Let n be the smallest integer for which $S(n) - S(5n) = 2013$. Determine the number of digits in n .

6 Let x, y, z be real numbers that are greater than or equal to 0 and less than or equal to $\frac{1}{2}$.

(a) Determine the minimum possible value of

$$x + y + z - xy - yz - zx$$

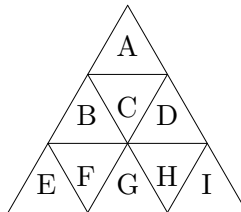
and determine all triples (x, y, z) for which this minimum is obtained.

(b) Determine the maximum possible value of

$$x + y + z - xy - yz - zx$$

and determine all triples (x, y, z) for which this maximum is obtained.

- 7 Consider the following layouts of nine triangles with the letters $A, B, C, D, E, F, G, H, I$ in its interior.



A sequence of letters, each letter chosen from $A, B, C, D, E, F, G, H, I$ is said to be *triangle-friendly* if the first and last letter of the sequence is C , and for every letter except the first letter, the triangle containing this letter shares an edge with the triangle containing the previous letter in the sequence. For example, the letter after C must be either A, B or D . For example, $CBFBC$ is triangle-friendly, but $CBFGH$ and $CBBHC$ are not.

- (a) Determine the number of triangle-friendly sequences with 2012 letters.
 (b) Determine the number of triangle-friendly sequences with exactly 2013 letters.
- 8 Let $\triangle ABC$ be an acute-angled triangle with orthocentre H and circumcentre O . Let R be the radius of the circumcircle.

Let A' be the point on AO (extended if necessary) for which $HA' \perp AO$.

Let B' be the point on BO (extended if necessary) for which $HB' \perp BO$.

Let C' be the point on CO (extended if necessary) for which $HC' \perp CO$.

Prove that $HA' + HB' + HC' < 2R$.

(Note: The orthocentre of a triangle is the intersection of the three altitudes of the triangle. The circumcircle of a triangle is the circle passing through the triangle's three vertices. The circumcentre is the centre of the circumcircle.)