

Canadian Mathematical Olympiad Qualification Repechage Official
Solutions

1 Determine all real solutions to the following equation:

$$2^{(2^x)} - 3 \cdot 2^{(2^{x-1}+1)} + 8 = 0.$$

Solution: Let $y = 2^{x-1}$. Then $2^x = 2y$. Then the left-hand side of the equation becomes

$$2^{2y} - 3 \cdot 2^{y+1} + 8 = 0.$$

Equivalently,

$$2^{2y} - 6 \cdot 2^y + 8 = 0.$$

This factors as

$$(2^y - 4)(2^y - 2) = 0.$$

Therefore, $2^y = 2$ or 4 . This yields the solutions $y = 1, 2$. Therefore, $2^{x-1} = 1, 2$, which yields solutions $x - 1 = 0, 1$. Hence, $x = 1, 2$.

We now verify these are indeed solutions. If $x = 1$, then $2^{2^1} - 3 \cdot 2^{2^0+1} + 8 = 2^2 - 3 \cdot 2^2 + 8 = 4 - 12 + 8 = 0$. Hence, $x = 1$ is a solution. If $x = 2$, then $2^{2^2} - 3 \cdot 2^{2^1+1} + 8 = 2^4 - 3 \cdot 2^3 + 8 = 16 - 24 + 8 = 0$. Hence, the solutions are indeed $x = 1, 2$. \square

2 In triangle ABC , $\angle A = 90^\circ$ and $\angle C = 70^\circ$. F is point on AB such that $\angle ACF = 30^\circ$, and E is a point on CA such that $\angle CFE = 20^\circ$. Prove that BE bisects $\angle B$.

Solution: By the angle bisector theorem, it suffices to show that

$$\frac{BA}{BC} = \frac{EA}{EC}.$$

By the definition of \sin , we have that $\sin 70^\circ = BA/BC$. By sine law,

$$\frac{EA}{EC} = \frac{EA}{EF} \cdot \frac{EF}{EC} = \sin 40^\circ \cdot \frac{\sin \angle ECF}{\sin \angle EFC} = \sin 40^\circ \cdot \frac{\sin 30^\circ}{\sin 20^\circ}.$$

Hence, it suffices to show that

$$\sin 70^\circ = \frac{\sin 40^\circ \sin 30^\circ}{\sin 20^\circ} = \frac{\sin 40^\circ}{2 \sin 20^\circ}.$$

By the double-angle formula for \sin , we have that $\sin 40^\circ = 2 \sin 20^\circ \cos 20^\circ$. Hence,

$$\frac{\sin 40^\circ}{2 \sin 20^\circ} = \cos 20^\circ = \sin 70^\circ.$$

This proves the desired equality. \square

- 3 A positive integer n has the property that there are three positive integers x, y, z such that $\text{lcm}(x, y) = 180$, $\text{lcm}(x, z) = 900$ and $\text{lcm}(y, z) = n$, where lcm denotes the lowest common multiple. Determine the number of positive integers n with this property.

Solution: Note that 5 divides into 180 only once. Hence, 5 divides into each of x, y at most once. But 5 divides into 900 twice, since 900 is divisible by 25. Since $\text{lcm}(x, z) = 900$ and 5 divides into x at most once, 5 divides into z exactly twice. Hence, z is divisible by 25.

Note that $900 = 2^2 \times 3^2 \times 5^2$ and z is a factor of 900 which is divisible by 5^2 . Therefore, z can be of the form

$$z = 2^a \times 3^b \times 5^2,$$

where $0 \leq a, b \leq 2$. Note also that since $\text{lcm}(x, y) = 180$, y is a factor of $180 = 2^2 \times 3^2 \times 5$. Therefore, y can only be of the form

$$y = 2^d \times 3^e \times 5^f,$$

where $0 \leq d, e \leq 2$ and $0 \leq f \leq 1$. Therefore, $n = \text{lcm}(y, z)$ must be of the form $2^r \times 3^s \times 5^2$, where $r = \max\{a, d\} \leq 2$ and $s = \max\{b, e\} \leq 2$. Therefore, $0 \leq r, s \leq 2$. I claim that all numbers of this form are feasible values of n . There are three choices of each of r, s , which yield in nine different possible values of n .

I claim that $(x, y, z) = (180, 1, n)$ satisfies the given equations. Clearly, $\text{lcm}(x, y) = \text{lcm}(180, 1) = 180$ and $\text{lcm}(x, z) = \text{lcm}(180, z) = \text{lcm}(2^2 \times 3^2 \times 5, 2^a \times 3^b \times 5^2) = 2^2 \times 3^2 \times 5^2 = 900$. Finally, $\text{lcm}(y, z) = \text{lcm}(1, n) = n$. This proves the claim.

Therefore, there are indeed nine possible values for n . \square

- 4 Four boys and four girls each bring one gift to a Christmas gift exchange. On a sheet of paper, each boy randomly writes down the name of one girl, and each girl randomly writes down the name of one boy. At the same time, each person passes their gift to the person whose name is written on their sheet. Determine the probability that *both* of these events occur:
- (i) Each person receives exactly one gift;
 - (ii) No two people exchanged presents with each other (i.e., if A gave his gift to B , then B did not give her gift to A).

Solution: The answer is $27/8192$.

Each of the eight persons has a choice of four people to give his/her gift to. Therefore, there are $4^8 = 2^{16}$ total number of combinations of how gifts can be exchanged.

Let A, B, C, D be the four boys and a, b, c, d be the four girls. There are $4! = 4 \times 3 \times 2 \times 1$ total number of ways for the four boys to give gifts to the girls so that each girl receives exactly one gift. Without loss of generality, suppose A gave a his gift, B gave b his gift, C gave c his gift and D gave d his gift.

Consider the boy girl a gave her gift to. Since A gave his gift to a , a did not give her gift to A . Hence, there are three boys for which girl a could have given her gift to. Without loss of generality, suppose a gave her gift to B . Note that b could not have given her gift to B , since B gave his gift to b . We now consider two cases:

If b gave her gift to A , then among A, B, a, b , we have the following cycle of exchanges: $A \rightarrow a \rightarrow B \rightarrow b \rightarrow A$. Then since C already gave his gift to c and D already gave his gift to d , then c must have given her gift to D and d must have given her gift to C . Hence, there is only one outcome in this case.

If b gave her gift to C , then we have the following sequence of exchanges so far: $A \rightarrow a \rightarrow B \rightarrow b \rightarrow C \rightarrow c$. Girl c could have given the gift to either A or D . But girl c could not have given her gift to A , since this would imply that D and d exchanged gifts. Therefore, girl c gave her gift to D and consequently, girl d gave her gift to A . The following is the resulting sequence of exchanges is $A \rightarrow a \rightarrow B \rightarrow b \rightarrow C \rightarrow c \rightarrow D \rightarrow d \rightarrow A$. This is the only possible outcome in this case.

If b gives her gift to D , then using the same argument as when b gives her gift to C , there is only one outcome in this case.

Hence, the total number of combination of exchanges that satisfy both (i) and (ii) is $24 \times 3 \times (1 + 1 + 1) = 2^3 \times 3^3$.

Therefore, the probability that both (i) and (ii) occur is $2^3 \times 3^3 / 2^{16} = 3^3 / 2^{13} = 27/8192$. \square

- 5 For each positive integer k , let $S(k)$ be the sum of its digits. For example, $S(21) = 3$ and $S(105) = 6$. Let n be the smallest integer for which $S(n) - S(5n) = 2013$. Determine the number of digits in n .

Solution: The answer is 504.

Given a digit A , define $f(A)$ to be the tens-digit of $5A$ and $g(A)$ be the ones-digit of $5A$. Note that

A	$f(A)$	$g(A)$
0	0	0
1	0	5
2	1	0
3	1	5
4	2	0
5	2	5
6	3	0
7	3	5
8	4	0
9	4	5

We will need the following lemma.

Lemma: Let n be a positive integer and let $A_{k-1}, A_{k-2}, \dots, A_0$ be the digits of n , from left to right. Then

$$S(n) - S(5n) = \sum_{j=0}^{k-1} (A_j - f(A_j) - g(A_j)).$$

Proof of Lemma: Clearly,

$$S(n) = \sum_{j=0}^{k-1} A_j.$$

We now consider $S(5n)$. Note that

$$n = \sum_{j=0}^{k-1} A_j 10^j.$$

Then

$$\begin{aligned} 5n &= \sum_{j=0}^{k-1} 5 \cdot A_j 10^j = \sum_{j=0}^{k-1} (10f(A_j) + g(A_j)) 10^j = \sum_{j=0}^{k-1} (f(A_j) \cdot 10^{j+1} + g(A_j) 10^j) \\ &= \sum_{j=0}^k (f(A_{j-1}) + g(A_j)) 10^j, \end{aligned}$$

where we define $f(A_{-1}) = 0$ and $g(A_k) = 0$. Note that $f(A_{j-1}) \in \{0, 1, 2, 3, 4\}$ and $g(A_j) \in \{0, 5\}$. Therefore, $f(A_{j-1}) + g(A_j) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, i.e. is a single digit. Therefore,

$$S(5n) = \sum_{j=0}^k (f(A_{j-1}) + g(A_j)) = \sum_{j=0}^{k-1} (f(A_j) + g(A_j)).$$

Therefore,

$$S(n) - S(5n) = \sum_{j=0}^{k-1} (A_j - f(A_j) - g(A_j)).$$

This proves the lemma. *End Proof of Lemma*

Given a digit A , define $f(A)$ to be the tens-digit of $5A$ and $g(A)$ be the ones-digit of $5A$. Note that

A	$A - f(A) - g(A)$
0	0
1	-4
2	1
3	-3
4	2
5	-2
6	3
7	-1
8	4
9	0

The value of $A - f(A) - g(A)$ is maximized when $A = 8$. Hence, to determine the minimum value of n such that $S(n) - S(5n) = 2013$, from the Lemma, we need to ensure that as many digits of n is 8 as possible. Since $8 - f(8) - g(8) = 4$, from the Lemma, n must have at least $\lceil 2013/4 \rceil = 504$ digits. We claim that this is indeed the minimum number of digits by constructing a positive integer n with 504 digits that satisfies $S(n) - S(5n) = 2013$. Each occurrence of 8 in n contributes a value of 4 to the expression $S(n) - S(5n)$. Since $2013 \equiv 1 \pmod{4}$, a value of 1 is unaccounted for. From the previous table, note that $2 - f(2) - g(2) = 1$. Hence, any positive integer n with 504 digits, consisting of one 2 and 503 8's, will satisfy $S(n) - S(5n) = 2013$.

Therefore, the answer is 504.

6 Let x, y, z be real numbers that are greater than or equal to 0 and less than or equal to $\frac{1}{2}$.

(a) Determine the minimum possible value of

$$x + y + z - xy - yz - zx$$

and determine all triples (x, y, z) for which this minimum is obtained.

(b) Determine the maximum possible value of

$$x + y + z - xy - yz - zx$$

and determine all triples (x, y, z) for which this maximum is obtained.

Solution to (a): Note that $x + y + z - xy - yz - zx = x(1 - y) + y(1 - z) + z(1 - x) \geq 0$ since $x, y, z \leq 1$ and $0 < 1 - x, 1 - y, 1 - z \leq 1$. This proves the inequality.

We now determine the equality case. For equality to hold, each of $x(1 - y), y(1 - z), z(1 - x)$ must equal 0. Note that $x(1 - y) = 0$ if and only if $x = 0$. (Note that $y \neq 1$ since $0 \leq y \leq 1/2$.) Similarly, $y(1 - z) = 0$ if and only if $y = 0$. $z(1 - x) = 0$ if and only if $z = 0$. Hence, equality holds if and only if $(x, y, z) = (0, 0, 0)$. \square

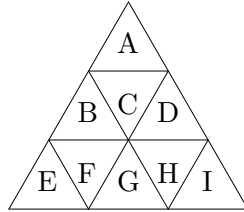
Solution 1 to (b): Let $a = 1/2 - x, b = 1/2 - y, c = 1/2 - z$. Note that $0 \leq a, b, c \leq 1/2$. Then $x + y + z - xy - yz - zx =$

$$3/2 - (a + b + c) - (3/4 - a - b - c + ab + bc + ca) = 3/4 - (ab + bc + ca) \leq 3/4,$$

since $a, b, c \geq 0$. Equality holds if and only if $ab = bc = ca = 0$. This holds if at least two of a, b, c are equal to zero. Therefore, equality holds in the original equation if and only if at least two of x, y, z equal $1/2$. \square

Solution 2 to (b): Let $S = x + y + z - xy - yz - zx$. Note that S is symmetric with respect to x, y, z . Hence, we may assume without loss of generality that $x \leq y \leq z$. We can rewrite S as $x(1 - y - z) + y + z - yz$. Note that $0 \leq 1 - y - z \leq 1$. If $y + z \neq 1$ and $x < 1/2$, then increasing x will strictly increase S . If $x = 1/2$, then since $x \leq y \leq z$, $(x, y, z) = (1/2, 1/2, 1/2)$, in which case $S = 3/4$. Finally, if $y + z = 1$, then $y = z = 1/2$, since $y, z \leq 1/2$. Hence, $S = y + z - yz = 3/4$. Therefore, in both cases, the maximum possible value of S is $3/4$. Hence, the maximum possible value of S is indeed $3/4$. In both cases, equality holds if and only if $y = z = 1/2$. Therefore, $S = 3/4$ if and only if two of x, y, z is equal to $1/2$. \square

- 7 Consider the following layouts of nine triangles with the letters $A, B, C, D, E, F, G, H, I$ in its interior.



A sequence of letters, each letter chosen from $A, B, C, D, E, F, G, H, I$ is said to be *triangle-friendly* if the first and last letter of the sequence is C , and for every letter except the first letter, the triangle containing this letter shares an edge with the triangle containing the previous letter in the sequence. For example, the letter after C must be either A, B or D . The sequence $CBFBC$ is triangle-friendly, but the sequences $CBFGH$ and $CBBHC$ are not.

- Determine the number of triangle-friendly sequences with exactly 2012 letters.
- Determine the number of triangle-friendly sequences with exactly 2013 letters.

Solution to (a): Color the triangles C, F, H red and the remaining triangles blue. Note that in any triangle-friendly sequence, the color of the letters alternate between red and blue. Since C is the first letter of any triangle-friendly sequence, the first letter of any triangle-friendly sequence is red. By parity, the odd numbered letters in such a sequence are red and the even numbered letters in such a sequence are blue. Therefore, the 2012th letter of any triangle-friendly sequence must be blue, and therefore cannot be C . Hence, there are zero triangle-friendly sequences with exactly 2012 letters.

Solution to (b): Define a triangle sequence to be a sequence with the same properties as a triangle-friendly sequence, but with the condition that the final letter can be any letter. Note that triangle-friendly sequences are also triangle sequences.

Using the same argument in (a), the odd-numbered terms of a triangle sequence must be C, F or H . Therefore, an odd-lengthed triangle sequence must end in C, F or H . For any non-negative integer n , let C_n, F_n, H_n be the number of triangle sequences of length $2n + 1$ that end of C, F, H , respectively. We need to determine C_{1006} . Note that $C_n + F_n + H_n$ is the total number of triangle sequences of length $2n + 1$.

By symmetry, note also that $F_n = H_n$ for all non-negative integer n .

We first determine the total number of triangle sequences of length $2n + 1$. We denote this quantity by T_n . Clearly, $T_0 = 1$, since C is the only triangle sequence of length 1. Inductively, given a triangle sequence of length $2n + 1$, the first $2n - 1$ letters form a triangle sequence of length $2n - 1$. By rotational symmetry, suppose the third last letter of the sequence is C . Then there are five ways to proceed from this letter, namely CAC, CBC, CDC, CBF, CDF . Hence, there are five times as many triangle sequences of length $2n + 1$ than triangle sequences of length $2n - 1$. Since $T_0 = 1$, $T_n = 5^n$.

We now determine recurrence relations between C_n, F_n, H_n . Consider a triangle sequence of length ≥ 3 , that ends in C . The third last letter of such a sequence is C, F, H . If this letter is C , then there are three ways, namely CAC, CBC, CDC , for the sequence to end in C . If this letter is F or H , then there is one way, namely FBC or HDC , for the sequence to end in C . Therefore,

$$C_n = 3C_{n-1} + F_{n-1} + H_{n-1} = 3C_{n-1} + 2F_{n-1}.$$

But we also know that $C_n + 2F_n = T_n = 5^n$. Therefore, $F_n = (5^n - C_n)/2$. Hence,

$$C_n = 3C_{n-1} + 2 \cdot \frac{5^{n-1} - C_{n-1}}{2} = 2C_{n-1} + 5^{n-1}.$$

Using this recurrence relation, with the initial condition $C_0 = 1$, we will determine a general formula for C_n . Since we need to determine the number of triangle-friendly sequences of length 2013, we need to determine C_{1006} .

We claim that

$$C_n = \frac{2^{n+1} + 5^n}{3},$$

for all non-negative integers n . We will prove this by induction on n . This is true for $n = 0$, since $(2^1 + 5^0)/3 = 1$, which is indeed equal to C_0 . Now suppose $C_m = \frac{2^{m+1} + 5^m}{3}$ for some non-negative integer m . Then

$$C_{m+1} = 2C_m + 5^m = 2 \left(\frac{2^{m+1} + 5^m}{3} \right) + 5^m = \frac{2^{m+2} + 2 \cdot 5^m + 3 \cdot 5^m}{3} = \frac{2^{m+2} + 5^{m+1}}{3},$$

which completes the induction proof.

Hence, the number of triangle-friendly sequences with length 2013 is

$$C_{1006} = \frac{2^{1007} + 5^{1006}}{3}.$$

8 Let $\triangle ABC$ be an acute-angled triangle with orthocentre H and circumcentre O . Let R be the radius of the circumcircle.

Let A' be the point on AO (extended if necessary) for which $HA' \perp AO$.

Let B' be the point on BO (extended if necessary) for which $HB' \perp BO$.

Let C' be the point on CO (extended if necessary) for which $HC' \perp CO$.

Prove that $HA' + HB' + HC' < 2R$.

(Note: The orthocentre of a triangle is the intersection of the three altitudes of the triangle. The circumcircle of a triangle is the circle passing through the triangle's three vertices. The circumcentre is the centre of the circumcircle.)

Solution: Since $\triangle ABC$ is an acute-angled triangle, H and O lie in the interior of $\triangle ABC$. By symmetry, we may assume without loss of generality that H lies in or on the boundary of $\triangle OBC$.

We will denote the area of a triangle XYZ by $[XYZ]$. Since H lies within or on $\triangle OBC$,

$$[OHB] + [OHC] \leq [OBC].$$

We now consider these three quantities. Note that

$$[OBC] = \frac{1}{2} \times OB \times OC \times \sin \angle BOC \leq \frac{1}{2} \times R \times R \times 1 = \frac{R^2}{2}.$$

Note also that

$$[OHB] = \frac{1}{2} \times OB \times HB' = \frac{1}{2} \times R \times HB'.$$

Similarly,

$$[OHC] = \frac{1}{2} \times R \times HC'.$$

Combining these three equations / inequalities yield

$$HB' + HC' \leq R.$$

Hence, to show that $HA' + HB' + HC' < 2R$, it suffices to show that $HA' < R$.

Note that $HA' \leq HO$, since A' is the foot of the perpendicular from H on OA . Since H lies in the interior of $\triangle ABC$, H also lies in the interior of the circumcircle of $\triangle ABC$, implying that OH is strictly shorter than the radius of the circumcircle of $\triangle ABC$. In other words, $HO < R$. Therefore, $HA' < R$.

This proves the inequality. \square