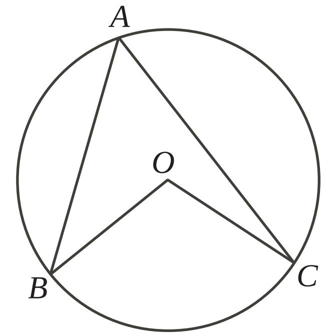
Problems

1. In the diagram, the circle has radius $\sqrt{7}$ and centre O. Points A, B and C are on the circle. If $\angle BOC = 120°$ and $AC = AB + 1$, determine the length of AB.



2. Brennan chooses a set $A = \{a, b, c, d, e\}$ of five real numbers with $a \le b \le c \le d \le e$. Delaney determines the subsets of A containing three numbers and adds up the numbers in these subsets. She obtains the sums $0, 3, 4, 8, 9, 10, 11, 12, 14, 19$. What are the five numbers in Brennan's set?

3. Determine all solutions to the system of equations:

$$\begin{aligned} x^2 + y^2 + x + y &= 12 \\ xy + x + y &= 3 \end{aligned}$$

4. Alphonse and Beryl play a game starting with a blank blackboard. Alphonse goes first and the two players alternate turns. On Alphonse's first turn, he writes the integer 10^{2011} on the blackboard. On each subsequent turn, each player can do exactly one of the following two things:

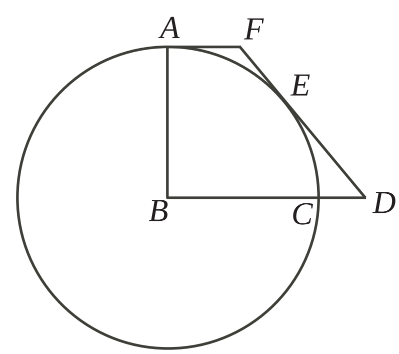
   (i) replace any number x that is currently on the blackboard with two integers a and b greater than 1 such that $x = ab$, or

   (ii) erase one or two copies of a number y that appears at least twice on the blackboard.

   Thus, there may be many numbers on the board at any time. The first player who cannot do either of these things loses. Determine which player has a winning strategy and explain the strategy.

5. Each vertex of a regular 11-gon is coloured black or gold. All possible triangles are formed using these vertices. Prove that there are either two congruent triangles with three black vertices or two congruent triangles with three gold vertices.

6. In the diagram, $ABDF$ is a trapezoid with AF parallel to BD and AB perpendicular to BD. The circle with centre B and radius AB meets BD at C and is tangent to DF at E. Suppose that x is equal to the area of the region inside quadrilateral $ABEF$ but outside the circle, that y is equal to the area of the region inside $\triangle EBD$ but outside the circle, and that $\alpha = \angle EBC$. Prove that there is exactly one measure α, with $0° \le \alpha \le 90°$, for which $x = y$ and that this value of α satisfies $\frac{1}{2} < \sin\alpha < \frac{1}{\sqrt{2}}$.



7. One thousand students participate in the 2011 Canadian Closed Mathematics Challenge. Each student is assigned a unique three-digit identification number abc, where each of a, b and c is a digit between 0 and 9, inclusive. Later, when the contests are marked, a number of markers will be hired. Each of the markers will be given a unique two-digit identification number xy, with each of x and y a digit between 0 and 9, inclusive. Marker xy will be able to mark any contest with an identification number of the form xyA or xAy or Axy, for any digit A. What is the minimum possible number of markers to be hired to ensure that all contests will be marked?

8. Determine all pairs (n, m) of positive integers for which there exists an infinite sequence $\{x_k\}$ of 0s and 1s with the properties that if $x_i = 0$ then $x_{i+m} = 1$ and if $x_i = 1$ then $x_{i+n} = 0$.