Problems

1. Suppose that a, b and x are positive real numbers. Prove that $\log_{ab} x = \dfrac{\log_a x \log_b x}{\log_a x + \log_b x}$.

2. Two tangents AT and BT touch a circle at A and B, respectively, and meet perpendicularly at T. Q is on AT, S is on BT, and R is on the circle, so that $QRST$ is a rectangle with $QT = 8$ and $ST = 9$. Determine the radius of the circle.

3. Prove that there is no real number x satisfying both equations
$$\begin{aligned} 2^x + 1 &= 2\sin x \\ 2^x - 1 &= 2\cos x \end{aligned}$$

4. Determine the smallest positive integer m with the property that $m^3 - 3m^2 + 2m$ is divisible by both 79 and 83.

5. The Fibonacci sequence is defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.
A Pythagorean triangle is a right-angled triangle with integer side lengths.
Prove that f_{2k+1} is the hypotenuse of a Pythagorean triangle for every positive integer k with $k \geq 2$.

6. There are 15 magazines on a table, and they cover the surface of the table entirely. Prove that one can always take away 7 magazines in such a way that the remaining ones cover at least $\frac{8}{15}$ of the area of the table surface.

7. If (a, b, c) is a triple of real numbers, define
   - $g(a, b, c) = (a + b, b + c, c + a)$, and
   - $g^n(a, b, c) = g(g^{n-1}(a, b, c))$ for $n \geq 2$.

   Suppose that there exists a positive integer n so that $g^n(a, b, c) = (a, b, c)$ for some $(a, b, c) \neq (0, 0, 0)$. Prove that $g^6(a, b, c) = (a, b, c)$.

8. Consider three parallelograms P_1, P_2, P_3. Parallelogram P_3 is inside parallelogram P_2, and the vertices of P_3 are on the edges of P_2. Parallelogram P_2 is inside parallelogram P_1, and the vertices of P_2 are on the edges of P_1. The sides of P_3 are parallel to the sides of P_1. Prove that one side of P_3 has length at least half the length of the parallel side of P_1.