(solutions follow)

1998-1999 Olympiad Correspondence Problems

Set 1

1. ABC is an isosceles triangle with $\angle A = 100^\circ$ and $AB = AC$. The bisector of angle B meets AC in D. Show that $BD + AD = BC$.

2. Let I be the incentre of triangle ABC. Let the lines AI, BI and CI produced intersect the circumcircle of triangle ABC at D, E and F respectively. Prove that EF is perpendicular to AD.

3. Let PQR be an arbitrary triangle. Points A, B and C external to the triangle are determine for which

$$\angle AQR = \angle ARQ = 15^\circ$$

$$\angle QPC = \angle RPB = 30^\circ$$

$$\angle PQC = \angle PRB = 45^\circ \ .$$

Prove that
(a) $AC = AB$;
(b) $\angle BAC = 90^\circ$.

4. Let a and b be two positive real numbers. Suppose that ABC is a triangle and D a point in side AC for which

$$\angle BCA = 90^\circ$$

$$|AD| = a \qquad\qquad |DC| = b \ .$$

Let $|BC| = x$ and $\angle ABD = \theta$. Determine the values of x and θ for the configuration in which θ assumes its maximum value.

5. Let $\mathcal{C}$ be a circle with centre O and radius k. For each point $P \neq O$, we define a mapping $P \longrightarrow P'$ where P' is that point on OP produced for which

$$|OP||OP'| = k^2 \quad .$$

In particular, each point on $\mathcal{C}$ remains fixed, and the mapping at other points has period 2. This mapping is called *inversion* in the circle $\mathcal{C}$ with centre O, and takes the union of the sets of circles and lines in the plane to itself. (You might want to see why this is so. Analytic geometry is one route.)

(a) Suppose that A and B are two points in the plane for which $|AB| = d$, $|OA| = r$ and $|OB| = s$, and let their respective images under the inversion be A' and B'. Prove that

$$|A'B'| = \frac{k^2 d}{rs} \quad .$$

(b) Using (a), or otherwise, show that there exists a sequence $\{X_n\}$ of distinct points in the plane with no three collinear for which all distances between pairs of them are rational.

6. Solve each of the following two systems of equations:

(a) $$x + xy + y = 2 + 3\sqrt{2} \ , \qquad x^2 + y^2 = 6;$$

(b)

$$x^2 + y^2 + \frac{2xy}{x+y} = 1 \ ,$$

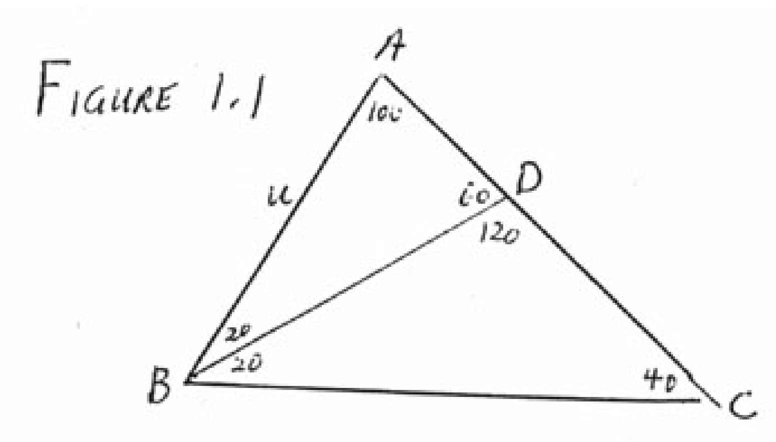
$$\sqrt{x+y} = x^2 - y \quad .$$

Solutions

Problem 1

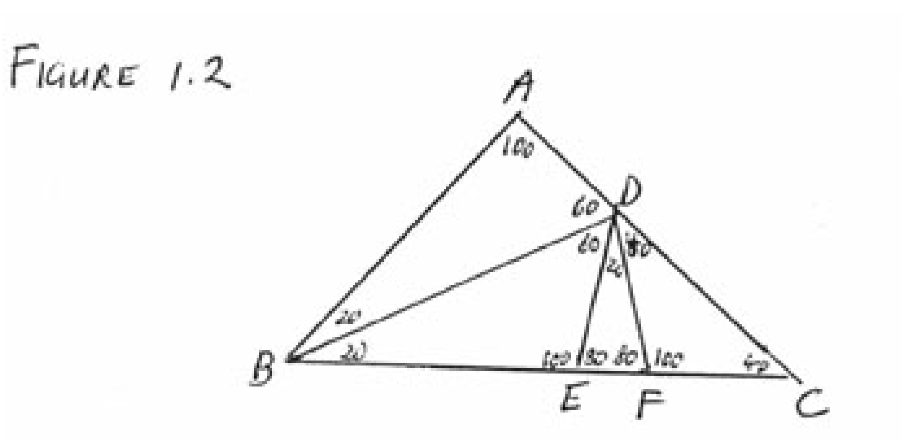
1. *First solution.* `See Figure 1.1.` Let $|AB| = u$. Then

$$\begin{aligned} |BD| + |AD| &= \frac{u \sin 100^\circ}{\sin 60^\circ} + \frac{u \sin 20^\circ}{\sin 60^\circ} \\ &= \frac{u}{\sin 60^\circ}(\sin 100^\circ + \sin 20^\circ) \\ &= \frac{2u \sin 60^\circ \cos 40^\circ}{\sin 60^\circ} = 2u \cos 40^\circ = |BC| \ . \end{aligned}$$



1. *Second solution.* `See Figure 1.2.` Let E and F be located in BC so that $\angle DEC = \angle DFB = 80^\circ$. Then $\Delta ABD \equiv \Delta EBD$ while triangles DEF, DCF and DBF are isosceles. Hence

$$BC = BF + FC = BD + DF = BD + DE = BD + AD \ .$$



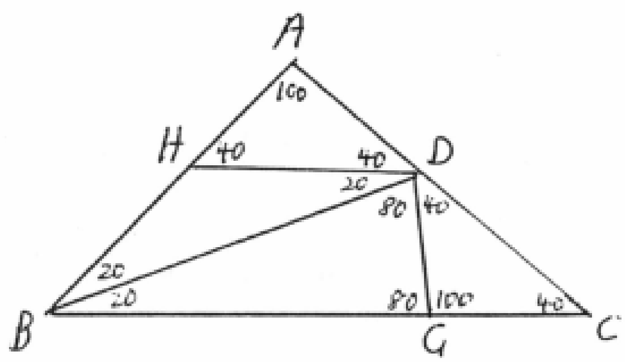
1. *Third solution.* `See Figure 1.3.` Select G on BC so that $BG = BD$ and H on AB so that $HD \| BC$. Then, by filling in the angles, we see that triangles BDG, HBD, AHD and DGC are isosceles. Also

triangles AHD and GCD are similar and $DC = BH = HD$. Hence $\Delta AHD \equiv \Delta GCD$ and $AD = GC$. Hence $BC = BG + GC = BD + AD$.

1. *Fourth solution.* `See Figure 1.3.` Define G as in the third solution. Triangles ABC and GCD are similar. Hence, using the angle bisection theorem, we have that $GC : CD = AB : BC = AD : DC$, so that $GC = AD$. Hence $BC = BG + GC = BD + AD$.

1. *Fifth solution.* `See Figure 1.3.` Define G and H as in the previous solution. $ABGD$ is concyclic, so $AD = DG$ since they are chords subtending equal angles of 20° at the circumference. It follows from the similarity of triangles AHD and GDC that they are congruent, so $AD = DG = GC$. Since also $BD = BG$, the result follows.

FIGURE 1.3



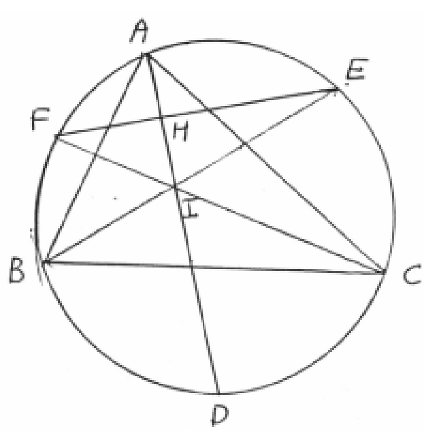
Problem 2.

2. *First solution.* `See Figure 2.1.` Let AD and EF meet at H. Then

$$\begin{aligned}\angle DHF &= \angle HEI + \angle HIE \\ &= \angle FEB + \angle AIE \\ &= \angle FCB + (\angle ABI + \angle BAI) \\ &= \frac{1}{2}\angle ACB + \frac{1}{2}\angle ABC + \frac{1}{2}\angle BAC = 90^\circ\end{aligned}$$

as desired.

FIGURE 2.1



2. *Second solution.* Identify points in the complex plane, with the circumcircle of ΔABC being the unit circle with centre 0. Let $A \sim 1$, $B \sim \cos 2\beta + i \sin 2\beta$, $C \sim \cos 2\gamma + i \sin 2\gamma$. Observe that AD bisects the arc BC, BE the arc CA and CF the arc AB, so that

$$D \sim \cos(\beta + \gamma) + i \sin(\beta + \gamma) \ ,$$

$$E \sim \cos(\gamma + \pi) + i \sin(\gamma + \pi) = -[\cos \gamma + i \sin \gamma] \ ,$$

$$F \sim \cos \beta + i \sin \beta \ .$$

The vector EF is given by

$$(\cos \beta + \cos \gamma) + i(\sin \beta + \sin \gamma) = \cos \frac{\beta - \gamma}{2} \left[\cos \frac{\beta + \gamma}{2} + i \sin \frac{\beta + \gamma}{2} \right]$$

and the vector AD by

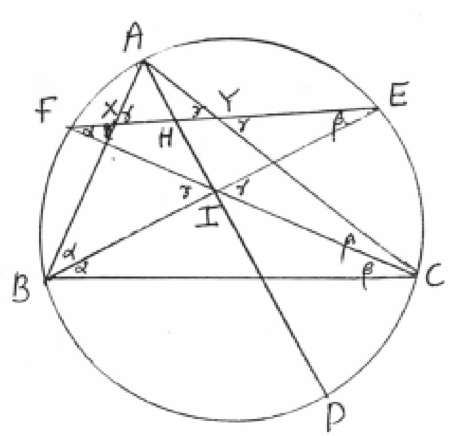
$$\begin{aligned}(\cos(\beta + \gamma) - 1) &+ i \sin(\beta + \gamma) \\ &= 2 \sin \frac{\beta + \gamma}{2} \left[- \sin \frac{\beta + \gamma}{2} + i \cos \frac{\beta + \gamma}{2} \right] \\ &= 2 \sin \frac{\beta + \gamma}{2} \left[\cos \frac{\beta + \gamma}{2} + i \sin \frac{\beta + \gamma}{2} \right] i\end{aligned}$$

from which it can be seen that they are perpendicular.

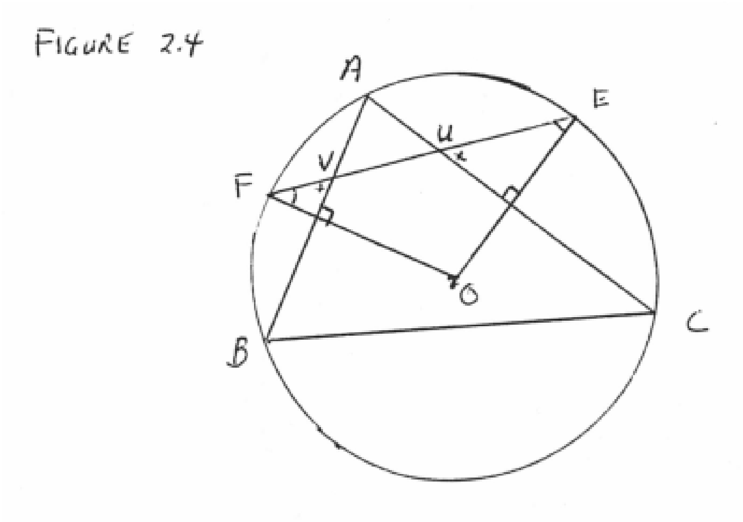
2. *Third solution.* [D. Brox] `See Figure 2.3.` Since $EFBC$ is concyclic, $\angle IFX = \angle CFE = \angle CBE = \angle EBA = \angle IBX$. Hence $FBIX$ is concyclic and so $\angle FXB = \angle FIB$. Similarly $\angle EYC = \angle EIC$.

   Hence $\angle AXY = \angle FXB = \angle FIB = \angle EIC = \angle EYC = \angle AYX$, so that $AX = XY$. Since ΔAXY is isosceles and AD bisects $\angle XAY$, then $AD \perp FE$.

FIGURE 2.3



2. *Fourth solution.* [L. Lessard] `See Figure 2.4.` Let O be the circumcentre of the triangle. Since F bisects arc AB, then OF right bisects AB. Also OE right bisects AC. Since ΔOEF is isosceles, $\angle OEF = \angle OFE$ and so $\angle EUC = \angle FVB$. Hence $\angle AUV = \angle AVU$, so that ΔAUV is isosceles. Thus, the bisector of angle A right bisects UV and the result follows.



2. *Fifth solution.* [H. Dong] `See Figure 2.3.` Using the fact that $AFBCE$ is concyclic, we have that $\angle AEF = \angle ACF = \angle BCF = \angle BEF$. Also $\angle AFE = \angle CFE$. Hence $\Delta AFE \equiv \Delta IFE$ (ASA) so that $EA = EI$. Thus ΔEAI is isosceles with apex angle AEI whose bisector EF must right bisect the base AI.

2. *Sixth solution.* `See Figures 2.6 and 2.3.` We first note a preliminary result: *If P, R, Q, S are four points on a circle and PQ and RS intersect inside the circle at T, then $\angle STQ = \angle SPQ + \angle PSR$, which is equal to half the sum of the angles subtended at the centre by arcs PR and SQ.* Now, $\angle ABE = \frac{1}{2}\angle ABC$, so that arc AE subtends an angle equal to $\angle ABC$ at the centre. Similarly BF subtends an angle equal to $\angle ACB$ at the centre and BD subtends an angle equal to $\angle BAC$ at the centre. Hence FD subtends an angle equal to $\angle ACB + \angle BAC$ at the centre. By the preliminary result, $\angle AHE$ is equal to half the sum of the angles subtended at the centre by arcs AE and FD, namely half of $180°$. The result follows.

Problem 3.

3. *First solution.* `See Figure 3.1.` (a) Let point D be selected on the same side of QR as A so that triangle QDR is equilateral. Then $DA \perp QR$ so that $\angle QDA = \angle RDA = 30°$ and $\angle DQA = \angle DRA = 45°$. Hence

$$\Delta PCQ \sim \Delta DAQ \qquad \text{and} \qquad \Delta PBR \sim \Delta DAR .$$

A rotation about Q followed by a dilatation takes C to A and P to D so that

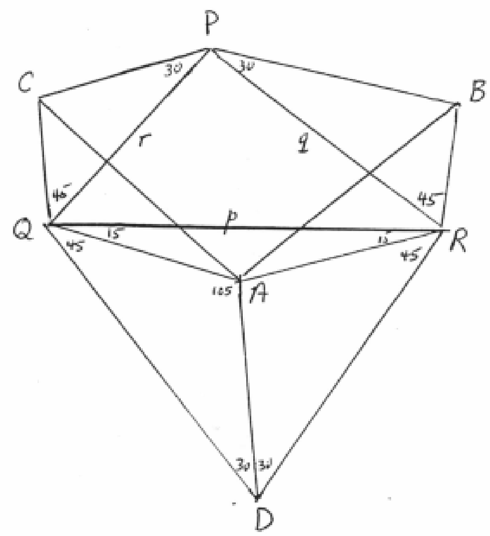
$$CQ : PQ = AQ : DQ .$$

Since $\angle CQA = \angle PQD$, $\Delta CAQ \sim \Delta PDQ$ so that $AC : PD = AQ : DQ$. Similarly, $\Delta BAR \sim \Delta PDR$ and $AB : PD = AR : DR$. Since $AQ = AR$ and $DQ = DR$, it follows that $AC = AB$.

(b) By the similar triangles identified in (a), $\angle CAQ = \angle PDQ$ and $\angle BAR = \angle PDR$. Hence

$$\begin{aligned}\angle CAB &= \angle QAR - (\angle CAQ + \angle BAR)\\ &= \angle QAR - (\angle PDQ + \angle PDR)\\ &= \angle QAR - \angle QDR = 150° - 60° = 90° .\end{aligned}$$

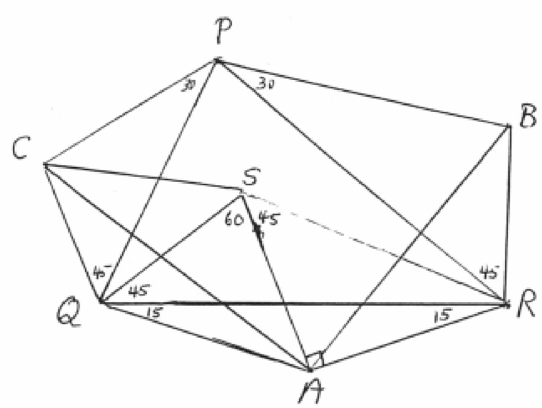
FIGURE 3.1



3. *Second solution.* `See Figure 3.2.` Let S be the image of R under a counterclockwise rotation about A through $90°$. Since $AS = AR$, $\angle ASR = \angle ARS = 45°$ so $\angle QRS = 30°$. Since $QA = AR = AS$, and since $\angle QAS = 150° - 90° = 60°$, $\angle AQS = \angle ASQ = 60°$ and so $\angle SQR = 45°$. Hence triangles CQP, BRP, SQR are similar, and $CQ : PQ = QS : QR$. Also $\angle CQS = 45° \pm \angle PQS = \angle PQR$ ($\pm$ according as S lies inside or outside of ΔPQR). Hence $\Delta CQS \sim \Delta PQR$. Therefore $\angle CSQ = \angle PRQ \Longrightarrow \angle CSA = \angle CSQ + 60° = \angle PRQ + 60° = \angle ARB$. Also $CS : RP = QC : QP = RB : RP \Longrightarrow CS = RB$. Since in addition $SA = RA$, $\Delta CSA \equiv \Delta BRA$ (SAS), so that $AC = AB$ and $\angle SAC = \angle RAB$. Finally,

$$\angle BAC = \angle BAS + \angle SAC = \angle BAS + \angle RAB = \angle RAS = 90° \ .$$

FIGURE 3.2



3. *Third solution.* [D. Brox] Note that $\sin 75° = \cos 15° = (\sqrt{6} + \sqrt{2})/4$ and $\sin 15° = (\sqrt{6} - \sqrt{2})/4$. Let

$$\alpha = \left(\frac{\sqrt{6} - \sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{3} - 1}{2} \ .$$

We solve the problem using vectors in the complex plane. Let lower case letters correspond to the points in the plane given in upper case, so that a corresponds to A, et cetera.

Since $|r-a| = |q-a| = |r-q|/2\cos 15^\circ = \frac{1}{2}(\sqrt{6}-\sqrt{2})|r-q|$, we have that

$$q-a = \frac{\sqrt{6}-\sqrt{2}}{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i\right)(q-r) = \alpha\left(\frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2}i\right)((q-p)-(r-p))$$

and

$$r-a = \frac{\sqrt{6}-\sqrt{2}}{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i\right)(r-q) = \alpha\left(\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i\right)((r-p)-(q-p)) \ .$$

Applying the sine law yields

$$|b-r| = |r-p|\sin 30^\circ/\sin 105^\circ = \frac{1}{2}(\sqrt{6}-\sqrt{2})|p-r| \quad \text{and} \quad |c-q| = \frac{1}{2}(\sqrt{6}-\sqrt{2})|q-p|$$

so that

$$b-r = \frac{\sqrt{6}-\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)(p-r) = \alpha(i-1)(r-p)$$

and

$$c-q = \frac{\sqrt{6}-\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)(p-q) = -\alpha(i+1)(q-p) \ .$$

Hence

$$\begin{aligned}
c-a = (q-a)+(c-q) &= \alpha\left[\left(\frac{\sqrt{3}-1}{2} - \frac{\sqrt{3}+1}{2}i\right)(q-p) - \left(\frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2}i\right)(r-p)\right] \\
&= i\alpha\left[-\left(\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i\right)(q-p) + \left(\frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i\right)(r-p)\right] \\
&= i[(r-a)+(b-r)] = i(b-a)
\end{aligned}$$

from which it follows that AC and AB are perpendicular segments of equal length.

3. *Fourth solution.* [D. Pritchard]
Lemma: For any triangle with angles α, β, γ,

$$\sin^2\alpha + \sin^2\gamma - 2\sin\alpha\sin\gamma\cos(\beta+60^\circ) = \sin^2\beta + \sin^2\gamma - 2\sin\beta\sin\gamma\cos(\alpha+60^\circ) \ .$$

Proof: Taking the difference between the two sides yields

$$\begin{aligned}
\sin^2\alpha - &\sin^2\beta - \sin\gamma[2\sin\alpha\cos(\beta+60^\circ) - 2\sin\beta\cos(\alpha+60^\circ)] \\
&= \frac{1}{2}[\cos 2\beta - \cos 2\alpha] - \sin(\alpha+\beta)[\sin(\alpha+\beta+60^\circ) + \sin(\alpha-\beta-60^\circ) \\
&\qquad - \sin(\beta+\alpha+60^\circ) - \sin(\beta-\alpha-60^\circ)] \\
&= \sin(\alpha+\beta)\sin(\alpha-\beta) - \sin(\alpha+\beta)[\sin(\alpha-\beta)\cos 60^\circ - \cos(\alpha-\beta)\sin 60^\circ \\
&\qquad - \sin(\beta-\alpha)\cos 60^\circ + \cos(\beta-\alpha)\sin 60^\circ] \\
&= \sin(\alpha+\beta)[\sin(\alpha-\beta) - 2\sin(\alpha-\beta)\cos 60^\circ] = 0 \ .
\end{aligned}$$

(a) Wolog, we can let the lengths of PQ, PR and QR be $\sin R$, $\sin Q$ and $\sin P$, respectively, since by the Law of Sines, these lengths are proportional to these quantities. Then

$$|AR| = \frac{\sin P\sin 15^\circ}{\sin 150^\circ} = \frac{\sqrt{3}-1}{\sqrt{2}}\sin P$$

$$|RB| = \frac{\sin Q \sin 30^\circ}{\sin 105^\circ} = \frac{\sqrt{3}-1}{\sqrt{2}} \sin Q$$

$$|AB|^2 = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 [\sin^2 P + \sin^2 Q - 2\sin P \sin Q \cos(R + 60^\circ)]$$

Similarly

$$|AC|^2 = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 [\sin^2 P + \sin^2 R - 2\sin P \sin R \cos(Q + 60^\circ)]$$

Hence $AB = AC$.

(b) $|BP| = \sin Q \sin 45^\circ / \sin 105^\circ = (\sqrt{3}-1)\sin Q$, $|CP| = (\sqrt{3}-1)\sin R$ and

$$|BC|^2 = (\sqrt{3}-1)^2[(\sin^2 R + \sin^2 Q - 2\sin R \sin Q \cos(P + 60^\circ)]$$

from which $|BC|^2 = 2|AB|^2$, and the result follows.

3. *Fifth solution.* [D. Nicholson] `See Figure 3.5.` Let $|AQ| = |AR| = u$, $|BR| = v$, $|CQ| = w$, $\alpha = \angle QPR$, $\beta = \angle PQR$ and $\gamma = \angle PRQ$. Let D, E, F be the respective feet of the perpendiculars from A, B, C to QR, PR and PQ.

$$|PQ| = |QC|\cos 45^\circ + |CF|\cot 30^\circ = \frac{1}{\sqrt{2}}(1+\sqrt{3})w$$

$$|PR| = \frac{1}{\sqrt{2}}(1+\sqrt{3})v$$

and

$$|QR| = 2u\cos 15^\circ = \frac{1}{\sqrt{2}}(1+\sqrt{3})u \ .$$

Hence

$$\cos\alpha = \frac{v^2 + w^2 - u^2}{2vw}$$

$$\cos\beta = \frac{u^2 + w^2 - v^2}{2uw}$$

$$\cos\gamma = \frac{u^2 + v^2 - w^2}{2uv} \ .$$

Then

$$\begin{aligned} |AC|^2 &= u^2 + w^2 - 2uw\cos(\beta + 60^\circ) \\ &= \frac{1}{2}(u^2 + v^2 + w^2) + \sqrt{3}uw\sin\beta \\ |AB|^2 &= u^2 + v^2 - 2uv\cos(\gamma + 60^\circ) \\ &= \frac{1}{2}(u^2 + v^2 + w^2) + \sqrt{3}uv\sin\gamma \ . \end{aligned}$$

Now $|PC| = |FC|\csc 30^\circ = 2|FC| = \sqrt{2}w$ and $|PB| = \sqrt{2}v$. Hence

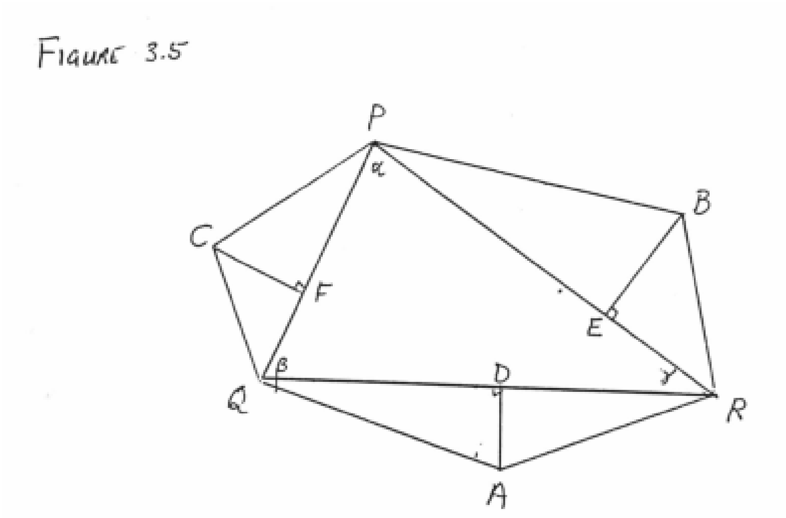
$$\begin{aligned} |BC|^2 &= 2v^2 + 2w^2 - 4vw\cos(\alpha + 60^\circ) \\ &= u^2 + v^2 + w^2 + 2\sqrt{3}vw\sin\alpha \ . \end{aligned}$$

Since

$$\sin\alpha : \sin\beta : \sin\gamma = |QR| : |PR| : |PQ| = u : v : w \ ,$$

$$uw\sin\beta = uv\sin\gamma = vw\sin\alpha \ .$$

Thus $|AC| = |AB|$ and $|BC|^2 = |AC|^2 + |AB|^2$, yielding the desired result.



3. *Sixth solution.* [T. Costin] `See Figure 3.1.` First, we establish a preliminary result. Let ABC be an arbitrary triangle. Using the standard notation and applying the sine law, we find that

$$a^2 + b^2 - 2ab\cos(C + 60^\circ) = \frac{b^2 \sin^2 A}{\sin^2 B} + b^2 - \frac{2b^2 \sin A}{\sin B}\left(\frac{\cos C}{2} - \frac{\sqrt{3}\sin C}{2}\right)$$
$$= \frac{b^2}{\sin^2 B}[\sin^2 A + \sin^2 B - \sin A \sin B \cos C + \sqrt{3}\sin A \sin B \sin C]$$
$$= \frac{b^2}{\sin^2 B}[\sin^2 B + \sin^2 A - \sin A \cos C(\sin A \cos C + \cos A \sin C) + \sqrt{3}\sin A \sin B \sin C]$$
$$= \frac{b^2}{\sin^2 B}[\sin^2 B + \sin^2 A \sin^2 C - \sin A \cos A \sin C \cos C + \sqrt{3}\sin A \sin B \sin C] \ .$$

Since this is symmetric in A and C, it is equal to $b^2 + c^2 - 2bc\cos(A + 60^\circ)$, and, by further symmetry, to $c^2 + a^2 - 2ac\cos(B + 60^\circ)$.

(a) Let D be constructed as in the figure, and note that $QC : QA = r : p$ by similar triangles. By the Law of Cosines,

$$AC^2 = QC^2 + AQ^2 - 2QC \cdot QA\cos(Q + 60^\circ)$$
$$= \frac{QA^2}{p^2}[r^2 + p^2 - 2pr\cos(Q + 60^\circ)]$$

and

$$AB^2 = \frac{RA^2}{p^2}[q^2 + p^2 - 2qp\cos(R + 60^\circ)] \ .$$

An application of the preliminary result to ΔPQR yields $AB = AC$.

(b) $PC : AD = r : p$ and $PB : AD = q : p$. Hence, by the Law of Cosines applied to ΔPCB,

$$BC^2 = \frac{AD^2}{p^2}[r^2 + q^2 - 2qr\cos(P + 60^\circ)] \ .$$

Since $p = QR = 2QA\cos 15^\circ$ and

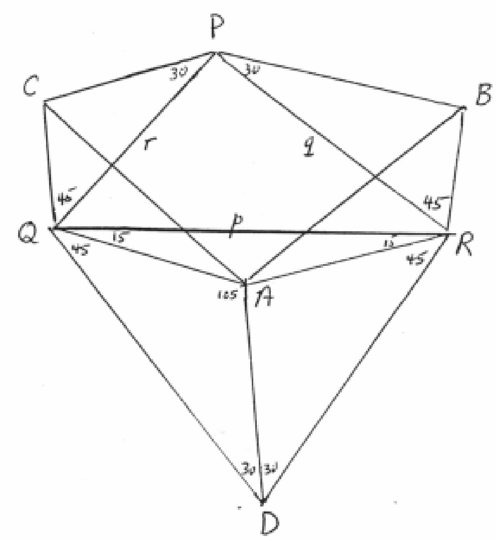
$$AD = QD\cos 30^\circ - QA\sin 15^\circ = p[\cos 30^\circ - \frac{1}{2}\tan 15^\circ]$$
$$= \frac{p}{2}[\sqrt{3} - (2 - \sqrt{3})] = (\sqrt{3} - 1)p \ ,$$

so $BC^2 = (4 - 2\sqrt{3})[r^2 + q^2 - 2qr\cos(P + 60°)]$. But

$$AB^2 + AC^2 = \frac{2QA^2}{p^2}[r^2 + p^2 - 2pr\cos(Q + 60°)] = \frac{1}{2}\sec^2 15°[r^2 + q^2 - 2qr\cos(P + 60°)] = BC^2$$

as desired.

FIGURE 3.1



3. *Seventh solution.* [Y. Shen] (a) From the Sine Law,

$$AR = AQ = QR(2\sin 15°)$$

$$BR = PR/2\sin 75°$$

and

$$CQ = PQ/2\sin 75° \ .$$

Hence

$$\begin{aligned} AB^2 &= AR^2 + BR^2 - 2AR \cdot BR\cos(60° + \angle PRQ) \\ &= 4\sin^2 15° QR^2 + \frac{PR^2}{2(1 - \cos 150°)} - 2RQ \cdot PR\tan 15° \left(\frac{1}{2} \cdot \frac{PR^2 + QR^2 - PQ^2}{2PR \cdot QR} - \frac{\sqrt{3}}{2} \cdot \sin\angle PRQ \right) \\ &= (2 - \sqrt{3})QR^2 + (2 - \sqrt{3})PR^2 - \frac{1}{2}(2 - \sqrt{3})(PR^2 + QR^2 - PQ^2) + (4\sqrt{3} - 6)[PQR] \\ &= \frac{1}{2}(2 - \sqrt{3})(PQ^2 + QR^2 + PR^2) + (4\sqrt{3} - 6)[PQR] \ . \end{aligned}$$

By symmetry, $AC^2 = AB^2$, so $AC = AB$.

(b)

$$BP = \frac{\sqrt{2}}{2} \cdot \frac{RP}{\sin 75°} \ , \qquad CP = \frac{\sqrt{2}}{2} \cdot \frac{PQ}{\sin 75°} \ .$$

Then

$$\begin{aligned} BC^2 &= BP^2 + CP^2 - 2BP \cdot CP\cos(60° + \angle QPR) \\ &= 2PR^2(2 - \sqrt{3}) + 2PQ^2(2 - \sqrt{3}) - (2 - \sqrt{3})(PR^2 + PQ^2 - QR^2) + (8\sqrt{3} - 12)[PQR] \\ &= AB^2 + AC^2 \end{aligned}$$

so that $\angle BAC = 90°$.

Problem 4.

4. *First solution.* `See Figure 4.1.` θ assumes its maximum value when the circumcircle of ΔADB is tangent to BC. For, if X is any other point on CB produced, X lies outside this circle and AD subtends at X a smaller angle than it subtends at the circumference of the circle.

Let O be the centre of the circumcircle and r its radius. Then O lies on the right bisector of AD and two radii are OD and OB. Since BC is tangent, $OB \perp BC$. Hence $x^2 + (a/2)^2 = r^2 = (b + (a/2))^2$ so that $x = \sqrt{b(a+b)}$. Since $\theta = \arctan \frac{a+b}{x} - \arctan \frac{b}{x}$,

$$\tan\theta = \frac{a/x}{1 + [b(a+b)/x^2]} = \frac{a}{2x} = \frac{a}{2\sqrt{b(a+b)}} \ .$$

Comment. The angle can also be identified by noting that

$$\angle ABD = \frac{1}{2}\angle AOD = \angle EOD = \arcsin\frac{(1/2)a}{r} = \arcsin\frac{a}{a+2b} \ .$$

4. *Second solution.* `See Figure 4.2.` (using calculus) As in (a), for general position of B we can calculate

$$\tan\theta = \frac{a/x}{1 + [b(a+b)/x^2]} = \frac{ax}{x^2 + b(a+b)} \ .$$

Since $0 < \theta < \pi/2$, θ is maximized when $\tan\theta$ is maximized or when $\cot\theta = 1/\tan\theta$ is minimized. Now

$$\begin{aligned}\frac{d}{dx}\cot\theta &= \frac{d}{dx}\left(\frac{x}{a} + \frac{b(b+a)}{ax}\right) \\ &= \frac{1}{a} - \frac{b(b+a)}{ax^2} \\ &= \frac{x^2 - b(a+b)}{ax^2} \ .\end{aligned}$$

Thus, $\cot\theta$ is decreasing when $x^2 < b(a+b)$ and increasing when $x^2 > b(b+a)$ and so achieves its minimum value of $(2\sqrt{b(a+b)})/a$ when $x = \sqrt{b(a+b)}$.

4. *Third solution.* [D. Brox] As in the previous solution we find that

$$\cot\theta = \frac{1}{a}\left[x + \frac{b(a+b)}{x}\right] \ .$$

Using the arithmetic-geometric means inequality, we find that $\cot\theta \geq \frac{2}{a}\sqrt{b(a+b)}$ with equality if and only if $x^2 = b(a+b)$. Hence the maximum value of $\theta = \operatorname{arccot}\,(\frac{2}{a}\sqrt{b(a+b)})$ is assumed when $x = \sqrt{b(a+b)}$.

4. *Fourth solution.* [K. Choi] Comparing two expressions for the area of triangle ABD, we have that

$$ax = \sin\theta\sqrt{x^2 + b^2}\sqrt{x^2 + (a+b)^2}$$

or

$$\begin{aligned}\csc^2\theta &= \frac{1}{a^2}\left[x^2 + (b^2 + (a+b)^2) + \frac{b^2(a+b)^2}{x^2}\right] \\ &= \frac{1}{a^2}\left[\left(x - \frac{b(a+b)}{x}\right)^2 + (a+2b)^2\right] \ .\end{aligned}$$

Since $0 < \theta < 90°$, θ is maximum when $\csc\theta$ is minimum, *i.e.*, when $x = \sqrt{b(a+b)}$. For this value of x,

$$\csc\theta = \frac{a+2b}{a} \quad \text{whence} \quad \theta = \arcsin\frac{a}{a+2b} \; .$$

4. *Fifth solution.* By the Law of Cosines,

$$a^2 = [(a+b)^2 + x^2] + [b^2 + x^2] - 2\sqrt{b^2(a+b)^2 + [a^2 + 2b(a+b)]x^2 + x^4}\cos\theta$$

whence

$$\cos^2\theta = \frac{[b(a+b) + x^2]^2}{[b(a+b) + x^2]^2 + a^2x^2}$$

and

$$\sin^2\theta = \frac{a^2x^2}{[b(a+b) + x^2]^2 + a^2x^2} \; .$$

Thus

$$\csc^2\theta = 1 + \left[\frac{b(a+b) + x^2}{ax}\right]^2 \; .$$

Since $0 < \theta < 90°$, to maximize θ, we must minimize $\csc^2\theta$, and hence minimize

$$\frac{1}{a}\left[\frac{b(a+b)}{x} + x\right] \; .$$

This can be done as in the third solution. When $x = \sqrt{b(a+b)}$, we find that

$$\sin^2\theta = \frac{a^2}{(2b+a)^2}$$

and so the maximum angle is $\arcsin(a/(2b+a))$.

4. *Sixth solution.* [Y. Shen] As in the second solution,

$$\tan\theta = \frac{ax}{x^2 + b(b+a)}$$

whence $(\tan\theta)x^2 - ax + b(b+a)\tan\theta = 0$. This quadratic is satisfiable by real values of x if and only if $a^2 \geq 4b(b+a)\tan^2\theta$ or

$$\tan\theta \leq \frac{a}{2\sqrt{b(b+a)}} \; .$$

When $\tan\theta = a/(2\sqrt{b(b+a)})$, then $x = \sqrt{b(b+a)}$, and we obtain the same result as before.

Problem 5.

5. *First solution.* `See Figure 5.1.` Since $\angle O$ is common and $OA : OB = OB' : OA'$, triangles OAB and $OB'A'$ are similar. Hence

$$r : s : d = \frac{k^2}{s} : \frac{k^2}{r} : |A'B'|$$

from which it follows that $|A'B'| = (k^2d)/(rs)$.

(b) We first show that the inversion with respect to a circle $\mathcal{C}$ of a line not passing through its centre O is a circle passing through O. Let F be the foot of the perpendicular from the point O to the line, and let P be any other point on the line. Let F' and P' be their respective images with respect to the inversion. As in (a), we have that $\Delta OPF \sim \Delta OF'P'$ whence $\angle OP'F' = \angle OFP = 90°$. Hence P' lies on the circle with diameter OF'.

Note that applying the inversion twice yields the identity, so that the image of F' is F. Let P' be any point distinct from O on the circle of diameter OF'. Its image P must satisfy $\angle OFP = \angle OP'F' = 90°$ and so it lies on the line perpendicular to OF. Hence the image of the line is the entire circle apart from O.

To construct our example, let $\mathcal{L}$ be a line at distance 1 from O, with F the foot of the perpendicular from O to the line. Let q be any rational number with $0 < q < 1$ and let P_q be selected on $\mathcal{L}$ with $\angle P_qOF = 2\arctan q$. Then $\tan \angle P_qOF = 2q(1-q^2)^{-1}$ so that $|FP_q| = 2q(1-q^2)^{-1}$ and $|OP_q| = (1+q^2)(1-q^2)^{-1}$. Hence all pairs of points P_q are rational distance apart and each P_q is a rational distance from O. Invert this line with respect to any circle with centre O and rational radius to obtain a circle through O. All images of points P_q lie on this circle, so no three are collinear. We can arrange these points in a sequence which satisfies the requirements.

5. *Second solution.* `See Figure 5.1.` (a) Suppose that $|A'B'| = d'$ and let $\angle AOB = \alpha$. By the Law of Cosines, $d^2 = r^2 + s^2 - 2rs\cos\alpha$. Since $|OA'| = k^2/r$ and $|OB'| = k^2/s$,

$$d'^2 = \frac{k^4}{r^2} + \frac{k^4}{s^2} - \frac{2k^4}{rs}\cos\alpha = \frac{k^4}{r^2s^2}[r^2 + s^2 - 2rs\cos\alpha] = \frac{k^4d^2}{r^2s^2}$$

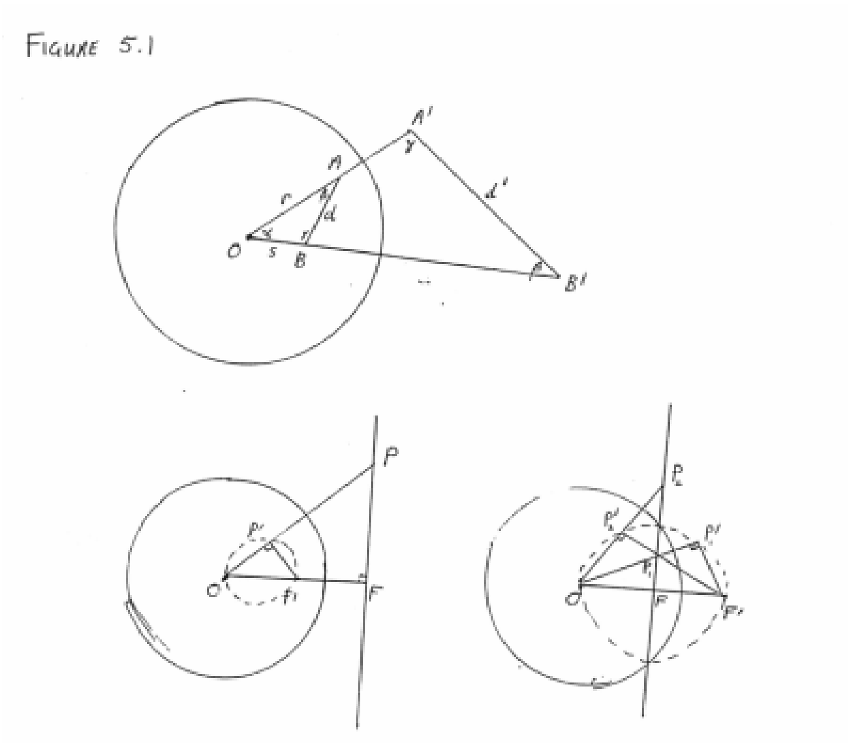
as desired. Observe that if k, r, s are rational, then d' is rational if and only if d is rational.

(b) In the cartesian plane, let $\mathcal{C}$ be the circle of radius 1 and centre $(0,0)$. Consider the line $\mathcal{L}$ with equation $x = 1$. A point (x,y) is on the image of this line under inversion with respect to $\mathcal{C}$ if and only if $x > 0$ and the point $(1, y/x)$ on the ray through $(0,0)$ and (x,y) satisfies $(x^2+y^2)(1+y^2/x^2) = 1$. This is equivalent to $x^2 + y^2 = x$ or $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$. Thus, the image of the line $\mathcal{L}$ is the circle of radius $\frac{1}{2}$ and centre $(\frac{1}{2}, 0)$, and no three points on this circle are collinear.

To solve the problem, we select a sequence $\{U_n\}$ of points on $\mathcal{L}$ for which $|OU_n|$ is rational and U_n has rational coordinates, and let $\{X_n\}$ be the images of these with respect to inversion in $\mathcal{C}$. But such a selection is possible since there are infinitely many rational pythagorean triples whose smallest number is 1. For example, we can take

$$U_n \sim \left(1, \frac{2n^2+2n}{2n+1}\right)$$

so that $|OU_n| = (2n^2+2n+1)/(2n+1)$.

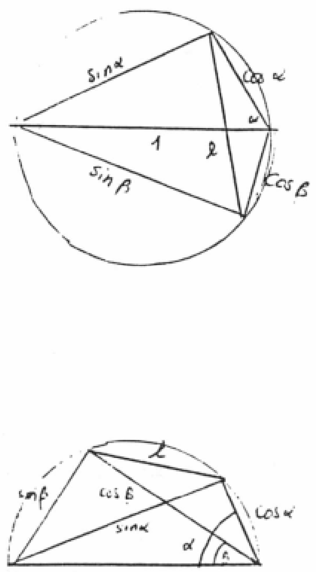


5. *Third solution.* [Y. Shen] An alternative approach to the solution of the second part comes by the use of Ptolemy's Theorem. `See Figure 5.3.` Let a circle of diameter l be given and two additional points on the circle be given whose chords make angles α and β with the diameter. If the points are on the same side of the diameter and are distant l apart, we have, by the Law of Cosines,

$$\begin{aligned} l^2 &= \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= \cos^2\alpha(1-\cos^2\beta) + \cos^2\beta(1-\cos^2\alpha) + 2\cos\alpha\cos\beta\sin\alpha\sin\beta) \\ &= (\cos\alpha\sin\beta + \sin\alpha\cos\beta)^2 \end{aligned}$$

so that $l = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. On the other hand, if the points are on the same side of the diameter, then $l^2 = \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta(\cos(\alpha-\beta))$ so that $l = \cos\alpha\sin\beta - \cos\beta\sin\alpha$. We can locate infinitely many points on the circumference of the circle for which the sine and cosine of the angle its chord makes with a given diameter are rational, and the distance between any pair of these points will be rational.

FIGURE 5 3



Problem 6.

6\. (a) *First solution.* Let $u = x + y$ and $v = xy$. Then $u + v = 2 + 3\sqrt{2}$ and $u^2 - 2v = 6$. Thus, $u^2 + 2u = 10 + 6\sqrt{2}$ so that $(u + 1)^2 = 11 + 6\sqrt{2} = (3 + \sqrt{2})^2$. Therefore

$$(u, v) = (2 + \sqrt{2}, 2\sqrt{2}) \quad \text{or} \quad (u, v) = (-4 - \sqrt{2}, 6 + 4\sqrt{2}) \ .$$

In the first case, x and y are roots of the quadratic equation

$$0 = t^2 - (2 + \sqrt{2})t + 2\sqrt{2} = (t - 2)(t - \sqrt{2})$$

so that $(x, y) = (2, \sqrt{2}), (\sqrt{2}, 2)$. In the second case, x and y are roots of the quadratic equation

$$0 = t^2 + (4 + \sqrt{2})t + (6 + 4\sqrt{2}) \ ,$$

and these are nonreal.

6\. (a) *Second solution.* [D. Nicholson] Observe that $(x+y+1)^2 = (x^2+y^2)+2(x+y+xy)+1 = 11+6\sqrt{2} = (3 + \sqrt{2})^2$. Thus $x + y = 2 + \sqrt{2}$ or $-4 - 2\sqrt{2}$, and we can proceed as in the first solution.

6\. (b) *First solution.* The first equation can be transformed to

$$(x + y)^2 - 2xy + \frac{2xy}{x + y} = 1$$

or to

$$\begin{aligned} 0 &= (x + y)^3 - (x + y) - 2xy[(x + y) - 1] \\ &= [(x + y) - 1][(x + y)^2 + (x + y) - 2xy] \\ &= [(x + y) - 1][(x^2 + y^2) + (x + y)] \ . \end{aligned}$$

Suppose that $x + y = 1$. Then the second equation becomes $x^2 - (1 - x) = 1$ or $0 = x^2 + x - 2 = (x + 2)(x - 1)$. Hence $(x, y) = (-2, 3), (1, 0)$ and both these solutions check out.

Suppose that $(x^2+y^2)+(x+y)=0$. Then x and y cannot be real. Otherwise, $x+y=-(x^2+y^2)\le 0$. By the first equation, $x+y\neq 0$ and by the second $x+y>0$ and we obtain a contradiction. From the second equation

$$\begin{aligned} x+y &= x^4-2x^2y+y^2 \\ \Longrightarrow 0 &= x^4-2yx^2-x+(y^2-y) \\ &= (x^2-x-y)(x^2+x-y+1) \end{aligned}$$

so that $y=x^2-x$ or $y=x^2+x+1$.

Let $x^2+y^2+x+y=0$ and $y=x^2-x$. Then $x^2=x+y=-(x^2+y^2)\Rightarrow y^2=-2x^2\Rightarrow y=\pm i\sqrt{2}x\Rightarrow x^2=(1\pm i\sqrt{2})x$. Since $x+y$ must be nonzero, $x=0$ is inadmissible. Hence $(x,y)=(1+i\sqrt{2},-2+i\sqrt{2}),(1-i\sqrt{2},-2-i\sqrt{2})$, and both of these check out.

Let $x^2+y^2+x+y=0$ and $y=x^2+x+1$. Then $y^2+2y-1=0$ or $(y+1)^2=2$. Hence $y=-1\pm\sqrt{2}$, so that $x^2+x+(2\mp\sqrt{2})=0$. Hence

$$x=\frac{-1\pm i\sqrt{7\pm 4\sqrt{2}}}{2}$$

and we obtain a pair of complex solutions for (x,y). Note that, when $y=x^2+x+1$, we have $x+y=(x+1)^2$ and so must take $\sqrt{x+y}=-(x+1)$ in order to corroborate the equation $\sqrt{x+y}=x^2-y$.

6. (b) *Second solution.* Let $x+y=u^2$ so that $u=x^2-y=x^2-(u^2-x)$, Therefore $0=x^2-u^2+(x-u)=(x-u)(x+u+1)$, whence $y=x^2-x$ or $y=x^2+x+1$.

Plugging $y=x^2-x$ into the first equation yields

$$0=x^4-2x^3+2x^2+2x-3=(x-1)(x+1)(x^2-2x+3)$$

so that

$$(x,y)=(1,0),(-1,2),(1+i\sqrt{2},-2+i\sqrt{2}),(1-i\sqrt{2},-2-i\sqrt{2})\ .$$

All of these work except $(x,y)=(-1,2)$ which is extraneous.

Plugging $y=x^2+x+1=(x+1)^2-x$ into the first equation and using $x+y=(x+1)^2$ yields

$$\begin{aligned} 0 &= x^2(x+1)^2+[(x+1)^2-x]^2(x+1)^2+2x[(x+1)^2-x]-(x+1)^2 \\ &= (x+1)^6-2x(x+1)^4+(2x^2+2x-1)(x+1)^2-2x^2 \\ &= [(x+1)^2-1][(x+1)^4-(2x-1)(x+1)^2+2x^2] \\ &= x(x+2)(x^4+2x^3+5x^2+4x+2)\ . \end{aligned}$$

The first two factors yield the possibilities $(x,y)=(0,1),(-2,3)$; the solution $(x,y)=(0,1)$ is extraneous, but $(x,y)=(-2,3)$ checks out. Since

$$x^4+2x^3+5x^2+4x+2=(x^2+x+2)^2-2=(y+1)^2-2\ ,$$

additional pairs of complex solutions can be obtained by solving

$$x^2+x+2=\pm\sqrt{2} \qquad y=-1\pm\sqrt{2}\ .$$

Comment. Since

$$x^4+2x^3+5x^2+4x+2=(x^2+x+1)^2+x^2+(x+1)^2=(x^2+x)^2+(2x+1)^2+1\ ,$$

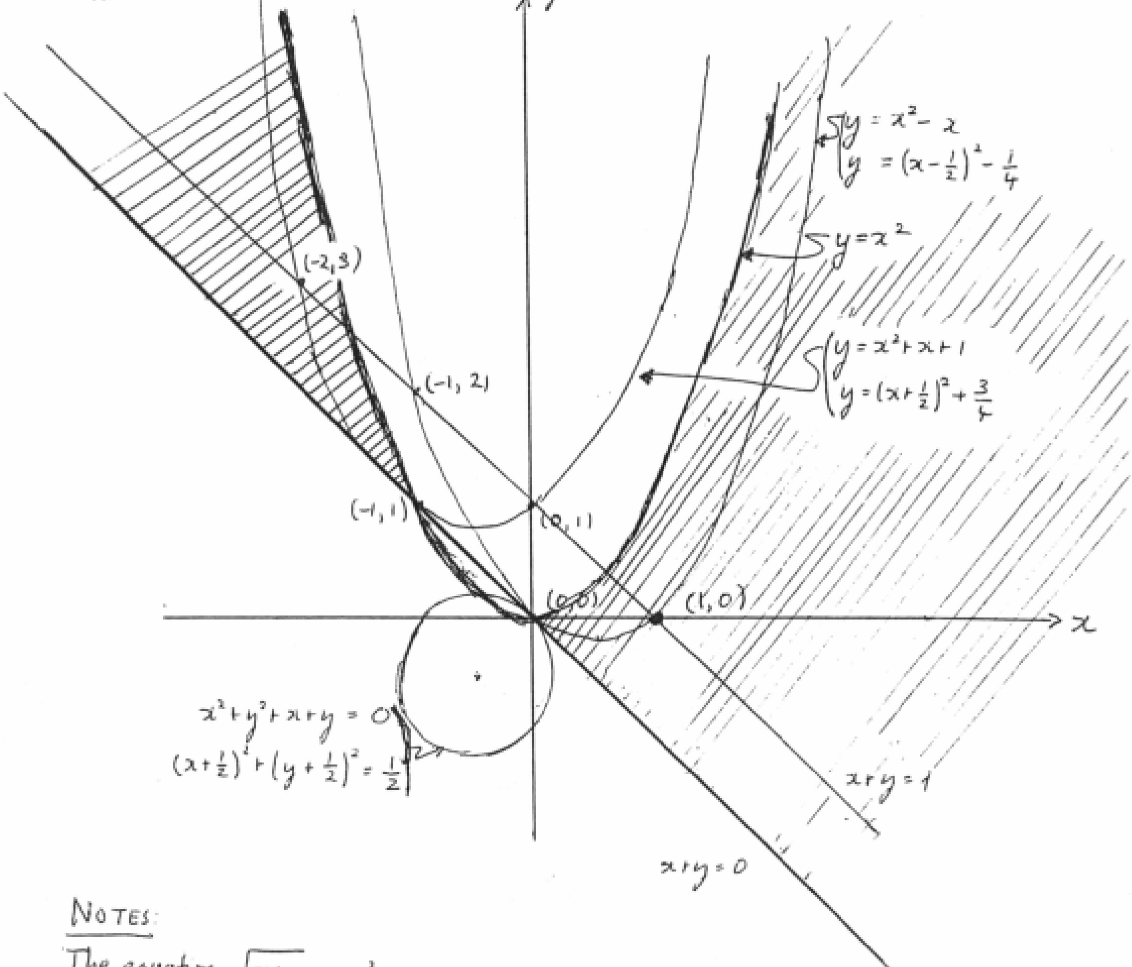
we see that the quartic factor will yield no more real solutions. By noting that

$$x^4+2x^3+5x^2+4x+2=(x^2+x)^2+4(x^2+x)+2\ ,$$

we can find these solutions by obtaining and then solving $x^2 + x = -2 \pm \sqrt{2}$.

We can get a picture of the situation for real solutions. `See Figure 6(b).` The equation $\sqrt{x+y} = x^2 - y$ requires $x + y \geq 0$ and $y \geq x^2$ to be viable for a real solution. The locus of this pair of inequalities is hatched. The first of the two given equations also requires $x + y \neq 0$. Hence there are no real solutions with $x^2 + y^2 + x + y = 0$. However, when $x + y = 1$ there is one real solution $(x, y) = (1, 0)$ for which $y = x^2 - x$ and one real solutions $(x, y) = (-2, 3)$ for which $y = x^2 + x + 1$. The putative solutions $(x, y) = (-1, 2), (0, 1)$ lie outside the hatched region and so are extraneous.

FIGURE 6(b)



NOTES:

The equation $\sqrt{x+y} = x^2 - y$ requires
$x + y \geq 0$ and $y \geq x^2$ to be viable for real solution.
The locus of this pair of inequalities is hatched.

Equation $x^2 + y^2 + \frac{2xy}{x+y} = 1$ further requires $x + y > 0$.

Hence, there are no real solutions with $x^2 + y^2 + x + y = 0$.

However, when $x + y = 1$ there is one real solution $(x, y) = (1, 0)$ for which $y = x^2 - x$
and one real solution $(x, y) = (-2, 3)$ for which $y = x^2 + x + 1$.

The putative solutions $(x, y) = (-1, 2), (0, 1)$ lie outside the hatched region and so are extraneous.