PROBLEMS FOR APRIL

Please send your solutions to

E.J. Barbeau
Department of Mathematics
University of Toronto
40 St. George Street
Toronto, ON M5S 2E4

no later than May 15, 2009.

Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

612. $ABCD$ is a rectangle for which $AB > AD$. A rotation with centre A takes B to a point B' on CD; it takes C to C' and D to D'. Let P be the point of intersection of the lines CD and $C'D'$. Prove that $CB' = DP$.

613. Let ABC be a triangle and suppose that

$$\tan \frac{A}{2} = \frac{p}{u} \qquad \tan \frac{B}{2} = \frac{q}{v} \qquad \tan \frac{C}{2} = \frac{r}{w} \ ,$$

where p, q, r, u, v, w are positive integers and each fraction is written in lowest terms.

(a) Verify that $pqw + pvr + uqr = uvw$.

(b) Let f be the greatest common divisor of the pair $(vw - qr, qw + vr)$, g be the greatest common divisor of the pair $(uw - pr, pw + ur)$, and h be the greatest common divisor of the pair $(uv - pq, pv + qu)$. Prove that

$$fp = vw - qr \qquad fu = qw + vr$$
$$gq = uw - pr \qquad gv = pw + ur$$
$$hr = uv - pq \qquad hw = pv + qu \ .$$

(c) Prove that the sides of the triangle ABC are proportional to $fpu : gqv : hrw$.

614. Determine those values of the parameter a for which there exist at least one line that is tangent to the graph of the curve $y = x^3 - ax$ at one point and normal to the graph at another.

615. The function $f(x)$ is defined for real nonzero x, takes nonzero real values and satisfies the functional equation

$$f(x) + f(y) = f(xyf(x + y)) \ ,$$

whenever $xy(x + y) \neq 0$. Determine all possibilities for f.

616. Let T be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the T?

617. Two circles are externally tangent at A and are internally tangent to a third circle Γ at points B and C. Suppose that D is the midpoint of the chord of Γ that passses through A and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that A is the incentre of triangle BCD.

618. Let a, b, c, m be positive integers for which $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$, and that there are actually possibilities with this value of m.

Challenge problems

The following problems are ones for which I do not have a solution. You are invited to solve them and send in your solution as early as you can. Four of the problems for this month were solved by students, whose solutions I acknowledge with thanks in order of receipt. Now they are for the rest of you to solve.

615. Hunter Spink (28/1/09); Jonathan Schneider (29/1/09)
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C5. Solve the equation
$$x^{12} - x^9 + x^4 - x = 1 \ .$$

C6. [Solved 2/3/09 by Jonathan Schneider] Suppose that $n > 1$ and that S is the set of all polynomials of the form
$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$
whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

C7. [Solved 2/3/09 by Jonathan Schneider and 4/3/09 by Cameron Bruggeman] Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial
$$p(z) = (z - a_1)^2(z - a_2)^2 \cdots (z - a_n)^2 + 1$$
cannot be written as the product of two nonconstant polynomials with integer coefficients.

C8. [Solved 2/3/09 by Jonathan Schneider] Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

C9. Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,
$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$
where z is the length of IP.

C10. Given the parameters a, b, c, solve the system
$$x + y + z = a + b + c;$$
$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

C11. Suppose that $x_i \geq 0$ and
$$\sum_{i=1}^{n} \frac{1}{1 + x_i} \leq 1 \ .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \ .$$

C12. Let $a_i \geq 2$,

$$x = \frac{1}{a_1} + \frac{1}{a_1^2 a_2} + \frac{1}{a_1^2 a_2^2 a_3} + \cdots \ ,$$

$$x = \frac{1}{a_1} + \frac{1}{a_1^3 a_2} + \frac{1}{a_1^3 a_2^3 a_3} + \cdots \ .$$

Prove that either x and y are both rational or x and y are both irrational.