Solutions to the June-July problems

388. A class with at least 35 students goes on a cruise. Seven small boats are hired, each capable of carrying 300 kilograms. The combined weight of the class is 1800 kilograms. It is determined that any group of 35 students can fit into the boats without exceeding the capacity of any one of them. Prove that it is unnecessary to leave any student off the cruise.

Solution. We prove the result by induction on the number of students. By hypothesis, if there are only 35 students, then all can be accommodated. Suppose that there are n students, where $n \geq 35$. Let the weights of these students be $w_1, w_2, \cdots, w_n$ kg in decreasing order.

Suppose that we have accommodated the k heaviest of these students for some $k \geq 35$. If $k < n$, then the weight w_{k+1} of the $(k+1)$th heaviest student satisfies

$$36w_{k+1} \leq w_1 + \cdots + w_{35} + w_{36} + \cdots + w_{k+1} \leq 1800 ,$$

whence $w_{k+1} \leq 50$. The amount of capacity available in the boats is

$$\begin{aligned} 2100 - (w_1 + \cdots + w_k) &\geq 2100 - (1800 - w_{k+1}) = 300 + w_{k+1} \\ &\geq 6w_{k+1} + w_{k+1} = 7w_{k+1} . \end{aligned}$$

Since all seven boats can accommodate at least $7w_{k+1}$ kg, at least one of them can accomodate at least w_{k+1} kg, and so the $(k+1)$th heaviest student can get into this boat. We can continue on in this way until all students are loaded.

389. Let each of m distinct points on the positive part of the $x-$axis be joined by line segments to n distinct points on the positive part of the $y-$axis. Obtain a formula for the number of intersections of these segments (exclusive of endpoints), assuming that no three of the segments are concurrent.

Solution 1. An intersection is determined by a choice of two points on each axis, and to each such choice there is exactly one intersection in which the segments are formed by taking the outer point on one axis and joining it to the inner point on the other. Thus there are $\binom{m}{2}\binom{n}{2}$ intersections.

Solution 2. [J. Park] Let the points on the $x-$axis be in order $X_1, \cdots, X_m$ and the points on the $y-$axis be in order $Y_1, \cdots, Y_n$. We draw the segments one at a time, starting with the segments $[X_i, Y_n]$ $(1 \leq i \leq m)$. This produces no interesection points. Now, for $1 \leq i \leq m$, draw $[X_i, Y_{n-1}]$, which produces $i-1$ intersection points with $[X_j, Y_n]$ when $1 \leq j \leq i-1$. All these segments ending in Y_{n-1} produce $1+2+\cdots+(m-1) = \binom{m}{2}$ intersections with segments ending in Y_n.

The segments ending in Y_{n-2} produce $\binom{m}{2}$ intersections with segments ending in Y_{n-1} and $\binom{m}{2}$ intersections with segments ending in Y_n. Continuing on, we find that the segments ending in Y_{n-j} make $j\binom{m}{2}$ intersections altogether with segments ending in Y_k for $k > n-j$ for $1 \leq j \leq n-1$. Hence the total number of intersections is

$$(1 + 2 + \cdots + \overline{n-1})\binom{m}{2} = \binom{n}{2}\binom{m}{2} .$$

390. Suppose that $n \geq 2$ and that $x_1, x_2, \cdots, x_n$ are positive integers for which $x_1 + x_2 + \cdots + x_n = 2(n+1)$. Show that there exists an index r with $0 \leq r \leq n-1$ for which the following $n-1$ inequalities hold:

$$x_{r+1} \leq 3$$

$$x_{r+1} + x_{r+2} \leq 5$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + r_{r+i} \leq 2i + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n \leq 2(n-r) + 1$$

$$\cdots$$

$$x_{r+1} + \cdots + x_n + x_1 + \cdots + x_j \leq 2(n+j-r) + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n + x_1 + \cdots + x_{r-1} \leq 2n - 1$$

where $1 \leq i \leq n-r$ and $1 \leq j \leq r-1$. Prove that, if all the inequalities are strict, then r is unique, and that, otherwise, there are exactly two such r.

Solution 1. First, consider the case that $n = 2$. Then $x_1 + x_2 = 6$. When $x_1 = 1, 2$, take $r = 0$; when $x_1 = 4, 5$, take $r = 1$, and when $x_1 = x_3 = 3$, take $r = 0$ or $r = 1$. Thus the result holds.

We prove the result by induction. Suppose that the result holds for $n = m \geq 2$. Let $x_1 + x_2 + x_3 + \cdots + x_{m+1} = 2(m+2)$. Observe that at least one x_i does not exceeds 2, since $3(m+1) > 2(m+2)$.

Case (i): Suppose that $x_k = 2$ for some k, which we may suppose exceeds 1. (Note that the conditions are cyclic in the indices.) Consider the set $\{x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{m+1}\}$. This satisfies the hypotheses for the case $n = m$. We may suppose that $r = 0$, as the argument is essentially the same for any index. Then

$$\sum_{i=1}^{s} x_i \leq 2s + 1 \qquad\qquad (1 \leq s \leq k-1)$$

$$\sum_{i=1}^{k-1} x_i + \sum_{i=k+1}^{s} x_i \leq 2(s-1) + 1 \qquad\qquad (k+1 \leq s \leq m) \ .$$

Restoring x_k, we find that $\sum_{i=1}^{s} x_i \leq 2s+1$ for $1 \leq s \leq m$. Thus, a solution for $n = m$ yields a solution for $n = m+1$. Conversely, any solution for $n = m+1$ yields a solution for $n = m$. If $r \neq k$, the same value of r works in both cases. If $r = k$, then taking out x_k will give a solution for the $n = m$ case with $r = k+1$. Equality will occur in the solution for $n = m$ if and only if it occurs in the solutions for $n = m+1$.

Case (ii): Suppose $x_k = 1$ and $x_{k+1} \geq 2$ (when $k = m+1$, interpret this as $x_1 \geq 2$). The set $\{x_1, x_2, \cdots, x_{k-1}, x_{k+1} - 1, \cdots, x_{m+1}\}$ satisfies the hypothesis for the $n = m$ case. We may suppose that $r = 0$. When $k \geq 2$,

$$\sum_{i=1}^{s} x_i \leq 2s + 1 \qquad\qquad (1 \leq s \leq k-1)$$

$$\sum_{i=1}^{k-1} x_i + \left(\sum_{i=k+1}^{s} x_i \right) - 1 \leq 2(s-1) + 1 \qquad (k+1 \leq s \leq m+1) \ .$$

Restoring x_k, we find that $\sum_{i=1} x_i \leq 2s+1$. Thus $r = 0$ continues to work for the $n = m+1$ case. However, when $k = 1$ and $r = 0$, we have that, when $2 \leq s \leq m-1$, $(\sum_{i=2}^{s} x_i) - 1 \leq 2s - 1$ so that $\sum_{i=2}^{s} x_i \leq 2s$. Hence $\sum_{i=1}^{s} x_i \leq 2s+1$ for $1 \leq i \leq m-1$. Equality will occur in the solution for $n = m$ if and only if it occurs in the solutions for $n = m+1$. The results follows.

Solution 2. [B.H. Deng] We establish the existence of the requisite value of r. If the inequalities hold for $r = 1$, Suppose that at least one inequality fails for $r = 1$. Then we can select $m \geq 2$ so that

(1) $x_2 + \cdots + x_m = 2(m-1) + w$, where $w \geq 2$;

(2) for $2 \leq i < m$, $x_2 + \cdots + x_i = 2(i-1) + u$ and $u < w$;

(3) for $m < j \leq n$, $x_2 + \cdots + x_j = 2(j-1) + v$ and $v \leq w$.

Thus, m is the first index where the discrepancy between $x_2 + \cdots + x_m$ and $2(m-1)$ achieves its maximum value.

We show that $r = m$. If $m < j \le n$, then

$$\begin{aligned} x_{m+1} + \cdots + x_j &= [2(j-1)+v] - [2(m-1)+w] = 2(j-m) + (v-w) \\ &\le 2(j-m) < 2(j-m)+1 \ . \end{aligned}$$

In particular,

$$\begin{aligned} x_{m+1} + \cdots + x_n &= (x_1 + \cdots + x_n) - (x_1 + \cdots + x_m) \\ &= 2(n+1) - x_1 - [2(m-1)+w] = 2(n-m) + 4 - x_1 - w \end{aligned}$$

from which

$$x_{m+1} + \cdots + x_n + x_1 = 2(n-m+1) + (2-w) \le 2(n-m+1) \ .$$

If $2 \le i < m$, then

$$\begin{aligned} x_{m+1} + \cdots + x_n + x_1 + \cdots + x_i &= 2(n-m) + 4 - w + 2(i-1) + u \\ &= 2(n-m+i) + 2 + (u-w) < 2(n-m+i) + 2 \ , \end{aligned}$$

from which

$$x_{m+1} + \cdots + x_n + x_1 + \cdots + x_i \le 2(n-m+i) + 1 \ .$$

We deal with the question of uniqueness. Wolog, we may suppose that the inequalities hold when $r = 1$. Suppose that $k \ge 2$ is such that $x_2 + \cdots + x_i \le 2(i-1)$ for $2 \le i < k$ and $x_2 + \cdots + x_k = 2(k-1)+1$. For $k < j \le n$, we have that

$$\begin{aligned} x_{k+1} + \cdots + x_j &= (x_2 + \cdots + x_j) - (x_2 + \cdots + x_k) \\ &\le 2(j-1) + 1 - 2(k-1) - 1 = 2(j-k). \end{aligned}$$

Also,

$$x_{k+1} + \cdots + x_n = 2(n+1) - x_1 - 2(k-1) - 1 = 2(n-k+1) + 1 - x_1$$

so that

$$x_{k+1} + \cdots + x_n + x_1 \le 2(n-k+1) + 1 \ .$$

For $2 \le i \le k-1$,

$$x_{k+1} + \cdots + x_n + x_1 + \cdots + x_i \le 2(n-k+1) + 1 + 2(i-1) = 2(n-k+i) + 1 \ ,$$

so that the inequalities hold for $r = k$.

Comment. When $n > 2$ and all the x_i exceed 1, then there is a direct argument. Since $3n > 2(n+1)$ and $x_1 + \cdots + x_n = 2(n+1)$, we must have $x_i \le 2$ for at least one index i. Since $x_i \ge 2$ for each i, there are two possibilities. Either, all the x_i but one are equal to 2 and the remaining one is equal to 4; or, all the x_i but two are equal to 2 and the remaining two are equal to 3. If $x_k = 4$, then we must take $r = k$ and we get strict inequality throughout.

In the second case, suppose, wolog, that $x_1 = 3$. If $x_2 = 3$ as well, then we can take $r = 1$ or $r = 2$; in the first case, $x_{r+1} = 3$, while in the second, $x_3 + \cdots + x_n + x_1 = 2(n-1) + 1 = 2n - 1$. If $x_k = 3$ for $3 \le k \le n-1$, then we can take $r = 1$ (in which case, for example, $x_2 + x_3 + \cdots + x_k = 2(k-1) + 1$) or $r = k$ (in which case, for example, $x_{k+1} + \cdots + x_1 = 2(n+1-k) + 1$.

391. Show that there are infinitely many nonsimilar ways that a square with integer side lengths can be partitioned into three nonoverlapping polygons with integer side lengths which are similar, but no two of which are congruent.

Comment. Unfortunately, there was an error in the formulation of the problem, and it is not known whether nonsimilar partitions can be made with integer side lengths as requested. However, full credit was given to any students who achieved infinitely many nonsimilar partitions, without satisfying the numerical condition.

Let $ABCD$ be the square. Let w be any integer and suppose that $u = 1+w$ and $v = 1+w+w^2$. Thus, $u(1+w+w^2) = v(1+w)$, and this will be the side length of the square. Select points E on AB, F on CD, G on DA and H on EF so that $|AE| = u$, $|EB| = uw(1+w)$, $|CF| = u(1+w)$, $|FD| = uw^2$, $|DG| = vw$, $|GA| = v$. Then the three trapezoids $AEHG$, $GHFD$ and $CFEB$ are similar and $|GH| = uw$. Different values of w give nonsimilar partitions, and the sides of the trapezoids parallel to the sides of the square will have integer lengths. For the slant sides to have integer lengths, it is necessary to make

$$u^2(w-1)^2 + v^2 = (w+1)^2(w-1)^2 + (w^2+w+1)^2 = 2w^4 + 2w^3 + w^2 + 2w + 2$$

a perfect square. The only possibility discovered so far is the case $w = 1$. We might achieve others if we can find values of m and n for which $w^2 - 1 = 2mn$ and $w^2 + w + 1 = m^2 - n^2$. It is unlikely that there are infinitely many possibilities. Can these by found using some other strategy?

392. Determine necessary and sufficient conditions on the real parameter a, b, c that

$$\frac{b}{cx+a} + \frac{c}{ax+b} + \frac{a}{bx+c} = 0$$

has exactly one real solution.

Solution 1. We look at a number of cases.

`Case 1.` $abc = 0$. If at least two of a, b, c vanish, then the equation is undefined. If, say, $a = 0$, $bc \neq 0$, then the equation becomes

$$\frac{b}{cx} + \frac{c}{b} = 0 \ ,$$

which, for $x \neq 0$ is equivalent to $c^2x + b^2 = 0$; this has the unique real solution $x = -b^2/c^2$.

`Case 2.` $abc \neq 0$, $(a^2 - bc)(b^2 - ca)(c^2 - ab) = 0$. Suppose, first, say, $a^2 - bc = b^2 - ca = 0$. Then, also, $c^2 - ab = 0$. It follows that $b/a = a/c = c/b = k$ for some nonzero value of k. Wew have that $k = c/b = (c/a)(a/b) = 1/k^2$, whence $k^3 = 1$ and $k = 1$. The equation becomes $0 = 3/(x+1)$ with no real solution.

Suppose, on the other hand, say, that $a^2 - bc = 0$ and $(b^2 - ca)(c^2 - ab) \neq 0$. Then $b/a = a/c = k$ for some real $k \neq 1$. Then $b/c = k^2$ and the equation becomes

$$0 = \left(\frac{b}{c} + \frac{c}{a}\right)\left(\frac{1}{x+k}\right) + \left(\frac{a}{c}\right)\left(\frac{1}{k^2x+1}\right) = \frac{g(x)}{k(x+k)(k^2x+1)}$$

where $g(x) = (k^5 + 2k^2)x + (2k^3 + 1)$. Now $g(-k) = -(k^3 - 1)(k^3 + 1)$ and $g(-1/k^2) = k^3 - 1$. If $k \neq -1$, then $g(x)$ is divisible by neither $(x+k)$ nor (k^2x+1) and the equation has a single real solution. If $k = -1$, then $b = c = -a$ and the equation reduces to $1/(x+1) = 0$ with no real solution.

`Case 3.` $abc(a^2 - bc)(b^2 - ca)(c^2 - ab)(ab^2 + bc^2 + ca^2) \neq 0$. For x such that $(cx+a)(ax+b)(bx+c) \neq 0$, the equation is equivalent to the quadratic equation $f(x) = 0$, where

$$\begin{aligned} f(x) &= b(ax+b)(bx+c) + c(cx+a)(bx+c) + a(cx+a)(ax+b) \\ &= (ab^2 + bc^2 + ca^2)x^2 + (a^3 + b^3 + c^3 + 3abc)x + (b^2c + c^2a + a^2b) \ . \end{aligned}$$

The discriminant of this quadratic is given by

$$\begin{aligned} D &\equiv (a^3 + b^3 + c^3 + 3abc)^2 - 4(ab^2 + bc^2 + ca^2)(b^2c + c^2a + a^2b) \\ &= a^6 + b^6 + c^6 - 3a^2b^2c^2 - 2(a^3b^3 + b^3c^3 + c^3a^3) + 2abc(a^3 + b^3 + c^3) \ . \end{aligned}$$

Since $f(-a/c) = bc^{-2}(a^2 - bc)(b^2 - ac) \neq 0$, $f(-b/a) \neq 0$ and $f(-c/b) \neq 0$, the equation $f(x) = 0$ has exactly the same solutions as the given equation. When $D = 0$, the equation has exactly one real solution. When $D \neq 0$, it has either no real solutions or exactly two real solutions.

`Case 4.` $abc(a^2 - bc)(b^2 - ca)(c^2 - ab)(a^3 + b^3 + c^3 + 3abc) \neq 0$; $ab^2 + bc^2 + ca^2 = 0$. In this case,

$$f(x) = (a^3 + b^3 + c^3 + 3abc)x + (b^2c + c^2a + a^2b)$$

and $f(x) = 0$ has exactly one real solution in the domain of the given equation.

`Case 5.` $abc(a^2 - bc)(b^2 - ca)(c^2 - ab) \neq 0$; $a^3 + b^3 + c^3 + 3abc = ab^2 + bc^2 + ca^2 = 0$. In this case, there is no real solution.

To sum up, we see that there is exactly one real solution if and only if one of the following conditions holds:

(a) exactly one of a, b, c vanishes;
(b) exactly two of b/a, a/c, c/b are equal to a real number distinct from -1 and $+1$;
(c) none of a, b, c, $a^2 - bc$, $b^2 - ac$, $c^2 - ab$, $ab^2 + bc^2 + ca^2$ vanishes and $D = 0$;
(d) none of a, b, c, $a^2 - bc$, $b^2 - ac$, $c^2 - ab$, $a^3 + b^3 + c^3 3abc$ vanishes and $ab^2 + bc^2 + ca^2 = 0$.

Comment. The difficulty of this problems lies in sorting out the different possibilities in order to make a comprehensive analysis. Most solvers simply put the left side of the equation over a common denominator and analyzed the quadratic equation with realizing what might happen with the coefficients. For example, one should be wary of the possibility that $a/b = b/c$, which means that $ax + b$ is a constant multiple of $bx + c$. In this case, putting the left side over a common cubic denominator introduces a spurious factor proportional to $ax + b$; the least common denominator is in fact no more than a quadratic.

Another take on the problem is to look at the graph of the left side. If the three linear denominators are not proportional, then there will be three vertical asymptotes, and one can analyze what the graph does between these asymptotes. This is a nice exercise for you to work on and show that the results are consistent with those obtained in the foregoing solution.

393. Determine three positive rational numbers x, y, z whose sum s is rational and for which $x - s^3$, $y - s^3$, $z - s^3$ are all cubes of rational numbers.

Solution 1. [A. Kong] Let $x = a/k$, $y = b/k$, $z = c/k$ where a, b, c, k are all integers. Suppose that $b = ma$ and $c = na$. Then

$$x - s^3 = \frac{1}{k^3}[ak^2 - (a+b+c)^3] = \frac{a}{k^3}[k^2 - a^2(1+m+n)^3]$$

$$y - s^3 = \frac{1}{k^3}[bk^2 - (a+b+c)^3] = \frac{a}{k^3}[mk^2 - a^2(1+m+n)^3]$$

$$z - s^3 = \frac{1}{k^3}[ck^2 - (a+b+c)^3] = \frac{a}{k^3}[nk^2 - a^2(1+m+n)^3] \ .$$

We try to make $x = s^3$, so that $k^2 = a^2(1+m+n)^3$; for example, let $a = 1$ and $1 + m + n$ be a square. When $(a, k, m, n) = (1, 8, 1, 2)$, we obtain the successful example $(x, y, z) = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4})$.

Solution 2. [C. Sun] Let $x - s^3 = a^3$, $y - s^3 = b^3$ and $z - s^3 = c^3$, so that $s - 3s^3 = a^3 + b^3 + c^3$. We try to make $a + b + c = x + y + z = s$; then we would have

$$\begin{aligned} s(2s-1)(2s+1) &= 4s^3 - s = s^3 - (s - 3s^3) \\ &= (a+b+c)^3 - (a^3 + b^3 + c^3) = 3(a+b)(b+c)(c+a) \ . \end{aligned}$$

Try $a+b = s$, $3(b+c) = 2s+1$ and $c+a = 2s-1$. Putting this with $a+b+c = s$ yields $c = (a+b+c)-(a+b) = 0$, $a = (c+a) - c = 2s - 1$ and $b = \frac{1}{3}(2s+1)$. Then

$$s = a + b + c = \frac{8s-2}{3} \Longrightarrow s = \frac{2}{5} \ .$$

This yields $(a,b,c) = (-\frac{1}{5}, \frac{3}{5}, 0)$ whence

$$(x,y,z) = (a^3+s^3, b^3+s^3, c^3+s^3) = \left(\frac{7}{125}, \frac{35}{125}, \frac{8}{125}\right) .$$

It can be checked that this works.

Solution 3. [F. Barekat] An example is

$$(x,y,z) = \left(\frac{49}{256}, \frac{49}{256}, \frac{62}{256}\right) .$$

To get this, let $x = p/u$, $y = q/v$ and $z = r/w$ and write s in the form m/uvw. Then

$$m^3 - pu^2v^3w^3 = \alpha^3 \ , \quad m^3 - qv^2u^3w^3 = \beta^3 \ , \quad m^3 - rw^2u^3v^3 = \gamma^3$$

which leads to $m(3m^2 - (uvw)^2) = \alpha^3 + \beta^3 + \gamma^3$. Playing around, we arrive at the possibility $m = 5$, $uvw = 8$ and $\alpha = \beta = 3$, $\gamma = 1$.

Solution 4. We try to make

$$x - s^3 = \frac{u^3s^3}{t^3} \ , \quad y - s^3 = \frac{v^3s^3}{t^3} \ , \quad z - s^3 = \frac{w^3s^3}{t^3} \ ,$$

which leads to

$$s - 3s^3 = s^3\left(\frac{u^3+v^3+w^3}{t^3}\right)$$

and

$$1 = s^2\left(\frac{u^3+v^3+w^3}{t^3} + 3\right) .$$

We select u, v, w to make the quantity in parentheses square (for example, we can try $u^3 + v^3 + w^3 = t^3$).

For example, $(u,v,w,t,s) = (3,4,5,6,1/2)$ yields

$$(8x, 8y, 8z) = \left(\frac{243}{216}, \frac{280}{216}, \frac{341}{216}\right)$$

and $(u,v,w,t,s) = (3,5,6,2,1/7)$ yields

$$(343x, 343y, 343z) = \left(\frac{35}{8}.\frac{133}{8}.\frac{224}{8}\right) .$$

These check out.

394. The average age of the students in Ms. Ruler's class is 17.3 years, while the average age of the boys is 17.5 years. Give a cogent argument to prove that the average age of the girls cannot also exceed 17.3 years.

Solution. Suppose that there are b boys and g girls and that the average age of the girls is a. Then the sum of all the ages of the students in the class is

$$17.3(b+g) = 17.5b + ag$$

whence $(17.3 - a)g = 0.2b$. Since the right side as well as g are positive, $17.3 - a$ must also be positive. Hence $a < 17.3$.