Solutions

318. Solve for integers x, y, z the system

$$1 = x + y + z = x^3 + y^3 + z^2 \ .$$

[Note that the exponent of z on the right is 2, not 3.]

Solution 1. Substituting the first equation into the second yields that

$$x^3 + y^3 + [1 - (x + y)]^2 = 1$$

which holds if and only if

$$\begin{aligned} 0 &= (x + y)(x^2 - xy + y^2) + (x + y)^2 - 2(x + y) \\ &= (x + y)(x^2 - xy + y^2 + x + y - 2) \\ &= (1/2)(x + y)[(x - y)^2 + (x + 1)^2 + (y + 1)^2 - 6] \ . \end{aligned}$$

It is straightforward to check that the only possibilities are that either $y = -x$ or $(x, y) = (0, -2), (-2, 0)$ or $(x, y) = (-3, -2), (-2, -3)$ or $(x, y) = (1, 0), (0, 1)$. Hence

$$(x, y, z) = (t, -t, 1), (1, 0, 0), (0, 1, 0), (-2, -3, 6), (-3, -2, 6), (-2, 0, 3), (0, -2, 3)$$

where t is an arbitrary integer. These all check out.

Solution 2. As in Solution 1, we find that either $x + y = 0, z = 1$ or $x^2 + (1 - y)x + (y^2 + y - 2) = 0$. The discriminant of the quadratic in x is

$$-3y^2 - 6y + 9 = -3(y + 1)^2 + 12 \ ,$$

which is nonnegative when $|y + 1| \leq 4$. Checking out the possibilities leads to the solution.

Solution 3.

$$\begin{aligned} (1 - z)(1 + z) = 1 - z^2 &= x^3 + y^3 \\ &= (x + y)[(x + y)^2 - 3xy] = (1 - z)[(1 - z)^2 - 3xy] \ , \end{aligned}$$

whence either $z = 1$ or $3xy = (1 - 2z + z^2) - (1 + z) = z(z - 3)$. The former case yields $(x, y, z) = (x, -x, 1)$ while the latter yields

$$x + y = 1 - z \qquad xy = \frac{1}{3}z(z - 3) \ .$$

Thus, we must have that $z \equiv 0 \pmod 3$ and that x, y are roots of the quadratic equation

$$t^2 - (1 - z)t + \frac{z(z - 3)}{3} = 0 \ .$$

The discriminant of this equation is $[12 - (z - 3)^2]/3$. Thus, the only possibilities are that $z = 0, 3, 6$; checking these gives the solutions.

319. Suppose that a, b, c, x are real numbers for which $abc \neq 0$ and

$$\frac{xb + (1 - x)c}{a} = \frac{xc + (1 - x)a}{b} = \frac{xa + (1 - x)b}{c} \ .$$

Is it true that, necessarily, $a = b = c$?

Comment. There was an error in the original formulation of this problem, and it turns out that the three numbers a, b, c are not necessarily equal. Note that in the problem, a, b, c, x all have the same status. Some solvers, incorrectly, took the given conditions as an identity in x, so that they assumed that the equations held for some a, b, c and all x.

Solution 1. Suppose first that $a+b+c \neq 0$. Then the three equal fractions are equal to the sum of their numerators divided by the sum of the denominators [why?]:

$$\frac{x(a+b+c)+(1-x)(a+b+c)}{a+b+c} = 1 \ .$$

Hence $a = xb+(1-x)c$, $b = xc+(1-x)a$, $c = xa+(1-x)b$, from which $x(b-c) = (a-c)$, $x(c-a) = (b-a)$, $x(a-b) = (c-b)$. Multiplying these three equations together yields that $x^3(b-c)(c-a)(a-b) = (a-c)(b-a)(c-b)$. Therefore, either $x = -1$ or at least two of a, b, c are equal.

If $x = -1$, then $a+b = 2c$, $b+c = 2a$ and $c+a = 2b$. This implies for example that $a-c = 2(c-a)$, whence $a = c$. Similarly, $a = b$ and $b = c$. Suppose on the other hand that, say, $a = b$; then $b = c$ and $c = a$.

The remaining case is that $a+b+c = 0$. Then each entry and sum of pairs of entries is nonzero, and

$$\frac{xa+(1-x)b}{-(a+b)} = \frac{x(-a-b)+(1-x)a}{b}$$

$$\Longrightarrow xab+(1-x)b^2 = x(a+b)^2-(1-x)(a^2+ab)$$

$$\Longrightarrow (1-x)(a^2+ab+b^2) = x(a^2+ab+b^2) \ .$$

Since $2(a^2+ab+b^2) = (a+b)^2+a^2+b^2 > 0$, $1-x = x$ and $x = 1/2$. But in this case, the equations become

$$\frac{b+c}{2a} = \frac{c+a}{2b} = \frac{a+b}{2c}$$

each member of which takes the value $-1/2$ for all a, b, c for which $a+b+c = 0$.

Hence, the equations hold if and only if either $a = b = c$ and x is arbitrary, or $x = 1/2$ and $a+b+c = 0$.

Comment. On can get the first part another way. If d is the common value of the three fractions, then

$$xb+(1-x)c = da \ ; \quad xc+(1-x)a = db \ ; \quad xa+(1-x)b = dc \ .$$

Adding these yeilds that $a+b+c = d(a+b+c)$, whence $d = 1$ or $a+b+c = 0$.

Solution 2. The first inequality leads to

$$xb^2+(1-x)bc = xac+(1-x)a^2$$

or

$$x(a^2+b^2)-x(a+b)c = a^2-bc \ .$$

Similarly

$$x(c^2+a^2)-x(c+a)b = b^2-ca \ ;$$

$$x(b^2+c^2)-x(b+c)a = c^2-ab \ .$$

Adding these three equations together leads to

$$2x[(a-b)^2+(b-c)^2+(c-a)^2] = (a-b)^2+(b-c)^2+(c-a)^2 \ .$$

Hence, either $a = b = c$ or $x = 1/2$.

If $x = 1/2$, then for some constant k,

$$\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c} = k \ ,$$

whence

$$-ka + b + c = a - kb + c = a + b - kc = 0 \ .$$

Add the three left members to get

$$(2-k)(a+b+c) = 0 \ .$$

Therefore, $k = 2$ or $a + b + c = 0$. If $k = 2$, then $a = b = c$, as in Solution 1. If $a + b + c = 0$, then $k = -1$ for any relevant values of a, b, c. Hence, either $a = b = c$ or $x = 1/2$ and $a + b + c = 0$.

320. Let L and M be the respective intersections of the internal and external angle bisectors of the triangle ABC at C and the side AB produced. Suppose that $CL = CM$ and that R is the circumradius of triangle ABC. Prove that

$$|AC|^2 + |BC|^2 = 4R^2 \ .$$

Solution 1. Since $\angle LCM = 90°$ and $CL = CM$, we have that $\angle CLM = \angle CML = 45°$. Let $\angle ACB = 2\theta$. Then $\angle CAB = 45° - \theta$ and $\angle CBA = 45° + \theta$. It follows that

$$\begin{aligned} |BC|^2 + |AC|^2 &= (2R \sin \angle CAB)^2 + (2R \sin \angle CBA)^2 \\ &= 4R^2(\sin^2(45° - \theta) + \sin^2(45° + \theta)) \\ &= 4R^2(\sin^2(45° - \theta) + \cos^2(45° - \theta)) = 4R^2 \ . \end{aligned}$$

Solution 2. [B. Braverman] $\angle ABC$ is obtuse [why?]. Let AD be a diameter of the circumcircle of triangle ABC. Then $\angle ADC = \angle CBM = 45° + \angle LCB$ (since $ABCD$ is concyclic). Since $\angle ACD = 90°$, $\angle DAC = 45° - \angle LCB = \angle CAB$. Hence, chords DC and CB, subtending equal angles at the circumference of the circumcircle, are equal. Hence

$$4R^2 = |AC|^2 + |CD|^2 = |AC|^2 + |BC|^2 \ .$$

321. Determine all positive integers k for which $k^{1/(k-7)}$ is an integer.

Solution. When $k = 1$, the number is an integer. Suppose that $2 \le k \le 6$. Then $k - 7 < 0$ and so

$$0 < k^{1/(k-7)} = 1/(k^{1/7-k}) < 1$$

and the number is not an integer. When $k = 7$, the expression is undefined.

When $k = 8$, the number is equal to 8, while if $k = 9$, the number is equal to 3. When $k = 10$, the number is equal to $10^{1/3}$, which is not an integer [why?].

Suppose that $k \ge 11$. We establish by induction that $k < 2^{k-7}$. This is clearly true when $k = 11$. Suppose it holds for $k = m \ge 11$. Then

$$m + 1 < 2^{m-7} + 2^{m-7} = 2^{(m+1)-7} \ ;$$

the desired result follows by induction. Thus, when $k \ge 11$, $1 < k^{1/(k-7)} < 2$ and the number is not an integer.

Thus, the number is an integer if and only if $k = 1, 8, 9$.

322. The real numbers u and v satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 \ .$$

Determine $u + v$.

Solution 1. The equations can be rewritten

$$u^3 - 3u^2 + 5u - 3 = 14 \ ,$$

$$v^3 - 3v^2 + 5v - 3 = -14 \ .$$

These can be rewritten as

$$(u-1)^3 + 2(u-1) = 14 \ ,$$

$$(v-1)^3 + 2(v-1) = -14 \ .$$

Adding these equations yields that

$$\begin{aligned} 0 &= (u-1)^3 + (v-1)^3 + 2(u+v-2) \\ &= (u+v-2)[(u-1)^2 - (u-1)(v-1) + (v-1)^2 + 2] \ . \end{aligned}$$

Since the quadratic $t^2 - st + s^2$ is always positive [why?], we must have that $u + v = 2$.

Solution 2. Adding the two equations yields

$$\begin{aligned} 0 &= (u^3 + v^3) - 3(u^2 + v^2) + 5(u + v) - 6 \\ &= (u+v)[(u+v)^2 - 3uv] - 3[(u+v)^2 - 2uv] + 5(u+v) - 6 \\ &= [(u+v)^3 - 3(u+v)^2 + 5(u+v) - 6] - 3uv(u+v-2) \\ &= \frac{1}{2}(u+v-2)[(u-v)^2 + (u-1)^2 + (v-1)^2 + 4] \ . \end{aligned}$$

Since the second factor is positive, we must have that $u + v = 2$.

Solution 3. [N. Horeczky] Since $x^3 - 3x^2 + 5x = (x-1)^3 + 2(x-1) + 3$ is an increasing function of x (since $x - 1$ is increasing), the equation $x^3 - 3x^2 + 5x - 17 = 0$ has exactly one real solution, namely $x = u$. But

$$\begin{aligned} 0 &= v^3 - 3v^2 + 5v + 11 \\ &= (v-2)^3 + 3(v-2)^2 + 5(v-2) + 17 \\ &= -[(2-v)^3 - 3(2-v)^2 + 5(2-v) - 17] \ . \end{aligned}$$

Thus $x = 2 - v$ satisfies $x^3 - 3x^2 + 5x - 17 = 0$, so that $2 - v = u$ and $u + v = 2$.

Comment. One can see also that each of the two given equations has a unique real root by noting that the sum of the squares of the roots, given by the cofficients, is equal to $3^2 - 2 \times 5 = -1$.

Solution 4. [P. Shi] Let m and n be determined by $u + v = 2m$ and $u - v = 2n$. Then $u = m + n$, $v = m - n$, $u^2 + v^2 = 2m^2 + 2n^2$, $u^2 - v^2 = 4mn$, $u^2 + uv + v^2 = 3m^2 + n^2$, $u^2 - uv + v^2 = m^2 + 3n^2$, $u^3 + v^3 = 2m(m^2 + 3n^2)$ and $u^3 - v^3 = 2n(3m^2 + n^2)$. Adding the equations yields that

$$\begin{aligned} 0 &= (u^3 + v^3) - 3(u^2 + v^2) + 5(u + v) - 6 \\ &= 2m^3 + 6mn^2 - 6m^2 - 6n^2 + 10m - 6 \\ &= 6(m-1)n^2 + 2(m^3 - 3m^2 + 5m - 3) \\ &= 6(m-1)n^2 + 2(m-1)(m^2 - 2m + 3) \\ &= 2(m-1)[3n^2 + (m-1)^2 + 2] \ . \end{aligned}$$

Hence $m = 1$.

323. Alfred, Bertha and Cedric are going from their home to the country fair, a distance of 62 km. They have a motorcycle with sidecar that together accommodates at most 2 people and that can travel at a maximum speed of 50 km/hr. Each can walk at a maximum speed of 5 km/hr. Is it possible for all three to cover the 62 km distance within 3 hours?

Solution 1. We consider the following regime. A begins by walking while B and C set off on the motorcycle for a time of t_1 hours. Then C dismounts from the motorcycle and continues walking, while B drives back to pick up A for a time of t_2 hours. Finally, B and A drive ahead until they catch up with C, taking a time of t_3 hours. Suppose that all of this takes $t = t_1 + t_2 + t_3$ hours.

The distance from the starting point to the point where B picks up A is given by

$$5(t_1 + t_2) = 50(t_1 - t_2)$$

km, and the distance from the point where B drops off C until the point where they all meet again is given by

$$5(t_2 + t_3) = 50(t_3 - t_2) \ .$$

Hence $45t_3 = 45t_1 = 55t_2$, so that $t_1 = t_3 = (11/9)t_2$ and so $t = (31/9)t_2$ and

$$t_1 = \frac{11}{31}t \ , \quad t_2 = \frac{9}{31}t \ , \quad t_3 = \frac{11}{31}t \ .$$

The total distance travelled in the t hours is equal to

$$50t_1 + 5(t_2 + t_3) = \frac{650}{31}$$

kilometers. In three hours, they can travel $1950/31 = 60 + (90/31) > 62$ kilometers in this way, so that all will reach the fair before the three hours are up.

Solution 2. Follow the same regime as in Solution 1. Let d be the distance from the start to the point where B drops C in kilometers. The total time for for C to go from start to finish, namely

$$\frac{d}{50} + \frac{62 - d}{5}$$

hours, and we wish this to be no greater than 3. The condition is that $d \geq 470/9$.

The time for B to return to pick up A after dropping C is $9d/550$ hours in which he covers a distance of $9d/11$ km. The total distance travelled by the motorcycle is

$$d + \frac{9d}{11} + (62 - \frac{2d}{11}) = \frac{18d + 682}{11}$$

km, and this is covered in

$$\frac{18d + 682}{550}$$

hours. To get A and B to their destinations on time, we wish this to not exceed 3; the condition for this is that $d \leq 484/9$. Thus, we can get everyone to the fair on time if

$$\frac{470}{9} \leq d \leq \frac{484}{9} \ .$$

Thus, if $d = 53$, for example, we can achieve the desired journey.

Solution 3. [D. Dziabenko] Suppose that B and C take the motorcycle for exactly $47/45$ hours while A walks after them. After $47/45$ hours, B leaves C to walk the rest of the way, while B drives back to pick up A. C reaches the destination in exactly

$$\frac{62-(47/45)50}{5}+\frac{47}{45}=3$$

hours. Since B and A start and finish at the same time, it suffices to check that that B reaches the fair on time. When B drops C off, B and A are 47 km apart. It takes B $47/55$ hours to return to pick up A. At this point, they are now

$$62-5\left(\frac{47}{45}+\frac{47}{55}\right)=62-47\left(\frac{20}{99}\right)=\frac{5198}{99}$$

km from the fair, which they will reach in a further

$$\frac{5198}{99\times 50}=\frac{2599}{2475}$$

hours. The total travel time for A and B is

$$\begin{aligned}&\frac{47}{45}+\frac{47}{55}+\frac{1}{50}\left[62-5\left(\frac{47}{45}+\frac{47}{55}\right)\right]\\&=\frac{9\times 47}{10\times 5}\left[\frac{1}{9}+\frac{1}{11}\right]+\frac{31}{25}=\frac{517+423+682}{550}=\frac{811}{275}\end{aligned}$$

hours. This is less than three hours.

324. The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with $|AB|=a$, $|BC|=b$ and $|VA|=|VB|=|VC|=|VD|=c$. Determine the area of the intersection of the pyramid and the plane parallel to the edge VA that contains the diagonal BD.

Solution 1. A dilation with centre C and factor $1/2$ takes A to S, the centre of the square and V to M, the midpoint of VC. The plane of intersection is the plane that contains triangle BMD. Since BM is a median of triangle BVC with sides c,c,b, its length is equal to $\frac{1}{2}\sqrt{2b^2+c^2}$ [why?]; similarly, $|DM|=\frac{1}{2}\sqrt{2a^2+c^2}$. Also, $|BD|=\sqrt{a^2+b^2}$. Let $\theta=\angle BMD$. Then, by the law of Cosines,

$$\cos\theta=\frac{c^2-a^2-b^2}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}}\ ,$$

whence

$$\sin\theta=\frac{\sqrt{4c^2(a^2+b^2)-(a^2-b^2)^2}}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}}\ .$$

The required area is

$$\frac{1}{2}|BM||DM|\sin\theta=\frac{1}{8}\sqrt{4c^2(a^2+b^2)-(a^2-b^2)^2}\ .$$

Comment. One can also use Heron's formula to get the area of the triangle, but this is more labourious. Another method is to calculate $(1/2)|BD||MN|$, where N is the foot of the perpendicular from M to BD, Note that, when $a\neq b$, N is not the same as S [do you see why?]. If $d=|BD|$ and $x=|SN|$ and, say $|MB|\leq|MD|$, then

$$|MN|^2=|MB|^2-\left(\frac{d}{2}-x\right)^2=|MD|^2-\left(\frac{d}{2}+x\right)^2$$

whence

$$x=\frac{|MD|^2-|MB|^2}{2d}\ .$$

If follows that

$$|MN|^2=\frac{2a^2b^2-a^4-b^4+4a^2c^2+4b^2c^2}{16(a^2+b^2)}\ .$$