PROBLEMS FOR NOVEMBER, 2004

Please send your solutions to
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no later than December 31, 2004. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

Notes. A real-valued function $f(x)$ of a real variable is *increasing* if and only if $u < v$ implies that $f(u) \leq f(v)$. The *circumcircle* of a triangle is that circle that passes through its three vertices; its centre is the *circumcentre* of the triangle. The *incircle* of a triangle is that circle that is tangent internally to its three sides; its centre is the *incentre* of the triangle.

346. Let n be a positive integer. Determine the set of all integers that can be written in the form

$$\sum_{k=1}^{n} \frac{k}{a_k}$$

where $a_1, a_2, \cdots, a_n$ are all positive integers.

347. Let n be a positive integer and $\{a_1, a_2, \cdots, a_n\}$ a finite sequence of real numbers which contains at least one positive term. Let S be the set of indices k for which at least one of the numbers

$$a_k, a_k + a_{k+1}, a_k + a_{k+1} + a_{k+2}, \cdots, a_k + a_{k+1} + \cdots + a_n$$

is positive. Prove that

$$\sum \{a_k : k \in S\} > 0 \ .$$

348. Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that $f(x) - x^3$ is an increasing function. Must $f(x) - x - x^2$ also be increasing?

349. Let s be the semiperimeter of triangle ABC. Suppose that L and N are points on AB and CB produced (*i.e.*, B lies on segments AL and CL) with $|AL| = |CN| = s$. Let K be the point symmetric to B with respect to the centre of the circumcircle of triangle ABC. Prove that the perpendicular from K to the line NL passes through the incentre of triangle ABC.

350. Let $ABCDE$ be a pentagon inscribed in a circle with centre O. Suppose that its angles are given by $\angle B = \angle C = 120^\circ$, $\angle D = 130^\circ$, $\angle E = 100^\circ$. Prove that BD, CE and AO are concurrent.

351. Let $\{a_n\}$ be a sequence of real numbers for which $a_1 = 1/2$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1} \ .$$

Prove that, for all n, $a_1 + a_2 + \cdots + a_n < 1$.

352. Let $ABCD$ be a unit square with points M and N in its interior. Suppose, further, that MN produced does not pass through any vertex of the square. Find the smallest value of k for which, given any position of M and N, at least one of the twenty triangles with vertices chosen from the set $\{A, B, C, D, M, N\}$ has area not exceeding k.