Problems for APRIL

Please send your solution to
Ed Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3

no later than June 15, 2004. It is important that your complete mailing address and your email address appear on the front page.

304. Prove that, for any complex numbers z and w,

$$(|z|+|w|)\left|\frac{z}{|z|}+\frac{w}{|w|}\right| \le 2|z+w| \ .$$

305. Suppose that u and v are positive integer divisors of the positive integer n and that $uv < n$. Is it necessarily so that the greatest common divisor of n/u and n/v exceeds 1?

306. The circumferences of three circles of radius r meet in a common point O. The meet also, pairwise, in the points P, Q and R. Determine the maximum and minimum values of the circumradius of triangle PQR.

307. Let p be a prime and m a positive integer for which $m < p$ and the greatest common divisor of m and p is equal to 1. Suppose that the decimal expansion of m/p has period $2k$ for some positive integer k, so that

$$\frac{m}{p} = .ABABABAB\ldots = (10^k A + B)(10^{-2k} + 10^{-4k} + \cdots$$

where A and B are two distinct blocks of k digits. Prove that

$$A + B = 10^k - 1 \ .$$

(For example, $3/7 = 0.428571\ldots$ and $428 + 571 = 999$.)

308. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by

$$f_0(x) \equiv 1$$
$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \ge 0$.

(a) Prove that, for all n, x,

$$f_n(x) = x^n f_n(1/x) \ .$$

(b) Determine a formula for the coefficient of x^k $(0 \le k \le n)$ in $f_n(x)$.

309. Let $ABCD$ be a convex quadrilateral for which all sides and diagonals have rational length and AC and BD intersect at P. Prove that AP, BP, CP, DP all have rational length.

310. (a) Suppose that n is a positive integer. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x+y)^{k-1}(y-k)^{n-k} \ .$$

(b) Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k} \ .$$