Solutions

213. Suppose that each side and each diagonal of a regular hexagon $A_1A_2A_3A_4A_5A_6$ is coloured either red or blue, and that no triangle $A_iA_jA_k$ has all of its sides coloured blue. For each $k = 1, 2, \cdots, 6$, let r_k be the number of segments A_kA_j $(j \neq k)$ coloured red. Prove that

$$\sum_{k=1}^{6}(2r_k - 7)^2 \leq 54 \ .$$

Solution 1. Suppose, say, $r_1 = 0$. Since every edge emanating from A_1 is blue, every other edge is red, so that $r_2 = r_3 = r_4 = r_5 = r_6 = 4$ and $\sum_{k=1}^{6}(2r_k - 7)^2 = 7^2 + 5 \times 1^2 = 54$.

Suppose, that every vertex is adjacent to at least one red edge, that, say, $r_1 = 1$ and that A_1A_2 is red. Then each of A_3, A_4, A_5, A_6 must be joined to each of the others by a red segment, so that r_3, r_4, r_5 and r_6 are at least equal to 3. Since all of them are joined to A_1 be a blue segment, r_3, r_4, r_5 and r_6 are at most equal to 4. Thus, $(2r_k - 7)^2 = 1$ for $3 \leq k \leq 6$. Since $1 \leq r_k \leq 5$, $\sum_{k=1}^{6}(2r_k - 7)^2 \leq 2 \times 5^2 + 4 \times 1^2 = 54$.

Suppose that $r_k \geq 2$ for each k. Then $2 \leq r_k \leq 5$, so that $(2r_k - 7)^2 \leq 3^2$ for each k and so $\sum_{k=1}^{6}(2r_k - 7)^2 \leq 6 \times 3^2 = 54$.

Solution 2. [A. Feiz Mohammadi] We prove the more general result: *Suppose that each side and each diagonal of a regular $n-$gon $A_1A_2 \cdots A_n$ is coloured either red or blue, and that no triangle $A_iA_jA_k$ has all of its sides coloured blue. For each $k = 1, 2, \cdots, n$, let r_k be the number of segments A_kA_j $(j \neq k)$ coloured red. Then*

$$\sum_{k=1}^{n}\left[2r_k - \left(\frac{3n-4}{2}\right)\right]^2 \leq \frac{n^3}{4} \ .$$

For $1 \leq k \leq n$, let b_k be the number of segments A_kA_j $(j \neq k)$ coloured blue. There are $\binom{b_k}{2}$ pairs of these segments; if A_kA_j and A_kA_i are two of them, then A_iA_j must be coloured red. Hence $\sum_{k=1}^{n}\binom{b_k}{2}$ counts the number of red segments, each as often as there are triangles containing it whose other edges are coloured blue. Suppose that A_uA_v is one of these red segments. There are b_u blue segments emanating from A_u and b_v from A_v, so that the red segments can be counted at most $\min\{b_u, b_v\} \leq \frac{1}{2}(b_u + b_v)$ times.

Hence

$$\sum_{k=1}^{n}\binom{b_k}{2} \leq \sum\left\{\frac{b_u + b_v}{2} : A_uA_v \ \text{ is coloured red } \right\} .$$

Each b_k will appear in r_k summands, and $r_k = (n-1) - b_k$, so that

$$\begin{aligned}
\frac{1}{2}\left[\sum_{k=1}^{n} b_k^2 - \sum_{k=1}^{n} b_k\right] = \sum_{k=1}^{n}\binom{b_k}{2} &\leq \frac{1}{2}\sum_{k=1}^{n} r_k b_k \\
&= \frac{1}{2}\sum_{k=1}^{n}[(n-1-b_k)b_k] = \frac{n-1}{2}\sum_{k=1}^{n} b_k - \frac{1}{2}\sum_{k=1}^{n} b_k^2 \\
&\Longrightarrow \sum_{k=1}^{n} b_k^2 \leq \frac{n}{2}\sum_{k=1}^{n} b_k \\
&\Longrightarrow \sum_{k=1}^{n}\left(2b_k - \frac{n}{2}\right)^2 \leq \frac{n^3}{4} \\
&\Longrightarrow \sum_{k=1}^{n}\left[2r_k - \left(\frac{3n-4}{2}\right)\right]^2 \leq \frac{n^3}{4} \ .
\end{aligned}$$

The upper bound in Feiz Mohammadi's result is actually attained when $r_1 = 0$ and $r_k = n-2$ for $k \geq 2$, and when $r_k = n-1$ for each k.

214. Let S be a circle with centre O and radius 1, and let P_i $(1 \leq i \leq n)$ be points chosen on the (circumference of the) circle for which $\sum_{i=1}^n \overrightarrow{OP_i} = \mathbf{0}$. Prove that, for each point X in the plane, $\sum |XP_i| \geq n$.

Solution 1. Use complex numbers, representing S by the unit circle in the complex plane and the points P_i by complex numbers z_i for which $|z_i| = 1$ and $\sum z_i = 0$. Then

$$\begin{aligned}\sum |z - z_i| &= \sum |z_i||z\overline{z_i} - 1| = \sum |z\overline{z_i} - 1| \\ &\geq \left|\sum (z\overline{z_i} - 1)\right| = \left|z(\sum \overline{z_i} - 1)\right| \\ &= \left|\overline{\sum z_i} - n\right| = |0 - n| = n\ .\end{aligned}$$

Solution 2. We have that

$$\begin{aligned}\sum |XP_i| &= \sum |\overrightarrow{OP_i} - \overrightarrow{OX}||\overrightarrow{OP_i}| \\ &\geq (\overrightarrow{OP_i - OX}) \cdot (\overrightarrow{OP_i}) \\ &= n - \sum \overrightarrow{OX} \cdot \overrightarrow{OP_i} \\ &= n - \overrightarrow{OX} \cdot \sum \overrightarrow{OP_i} = n\ .\end{aligned}$$

(The inequality is due to the Cauchy-Schwarz Inequality.)

Solution 3. [O. Bormashenko] Let the points $P_i \sim (\cos u_i, \sin u_i)$ be placed on the unit circle of the cartesian plane and let $X \sim (x, y)$. For $1 \leq i \leq n$,

$$\begin{aligned}(x \sin u_i - y \cos u_i)^2 \geq 0 &\Longleftrightarrow x^2 \sin^2 u_i + y^2 \cos^2 u_i \geq 2xy \cos u_i \sin u_i \\ &\Longleftrightarrow x^2 + y^2 \geq x^2 \cos^2 u_i + 2xy \cos u_i \sin u_i + y^2 \sin^2 u_i\ ,\end{aligned}$$

so that

$$\begin{aligned}|XP_i|^2 &= (x - \cos u_i)^2 + (y - \sin u_i)^2 \\ &= x^2 + y^2 + 1 - 2x \cos u_i - 2y \sin u_i \\ &\geq (1 - x \cos u_i - y \sin u_i)^2\ .\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=1}^n |XP_i| &\geq \sum_{i=1}^n (1 - x \cos u_i - y \sin u_i) \\ &= n - x \sum_{i=1}^n \cos u_i - y \sum_{i=1}^n \sin u_i = n\ ,\end{aligned}$$

because of $\sum_{i=1}^n \overrightarrow{OP_i} = \mathbf{O}$ and the vanishing of the components of this sum in the two coordinate directions.

Solution 4. [A. Mao] Let the equation of the circle S in the cartesian plane be $x^2 + y^2 = 1$. Wolog, we may assume that X lies on the $x-$axis. Let r and s be the lines of equations $x = 1$ and $x = -1$ respectively. If X lies outside the circle, the reflection in the nearer of the lines r and s take X to a point Y for which

$$|OY| = \begin{cases} 2 - |OX|, & \text{for } 1 < |OX| \leq 2; \\ |OX| - 2, & \text{for } |OP| \geq 2. \end{cases}$$

Since Y lies on the same side of the line of reflection as all of the P_i and X lies on the opposite side, $\sum |XP_i| \geq \sum |YP_i|$.

If $1 \leq |OX| < 3$, the first reflection takes X to the interior of the circle. If $|OX| \geq 3$, the first reflection reduces the distance from the origin by 2 and a chain of finitely many reflections will take X into the circle.

Hence, wolog, we may suppose that X lies within or on the circle. Let $X \sim (w, 0)$ with $-1 \leq w \leq 0$ and let $P_i \sim (\cos u_i, \sin u_i)$. Then

$$\begin{aligned} |XP_i| &= \sqrt{(w - \cos u_i)^2 + \sin^2 u_i} \\ &= \sqrt{w^2 - 2w\cos u_i + 1} \\ &= \sqrt{(1 - w\cos u_i)^2 + w^2 \sin^2 u_i} \\ &\geq \sqrt{(1 - w\cos u_i)^2} = 1 - w\cos u_i \ , \end{aligned}$$

since $|w \cos u_i| \leq 1$. Hence

$$\sum |XP_i| \geq n - w \sum \cos u_i = n \ .$$

215. Find all values of the parameter a for which the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four real solutions which are in geometric progression.

Solution 1. Let $x + (1/x) = t$. Then the equation becomes $f(t) \equiv 16t^2 - at + 2a - 15 = 0$. If the original equation has all real roots, then this quadratic in t must have two real roots t_1 and t_2, both of which have absolute value exceeding 2 (why?). The discriminant of the quadratic is equal to $a^2 - 64(2a - 15) = (a - 8)(a - 120)$, so that its roots are real if and only if $a \leq 8$ or $a \geq 120$. Observe that $f(2) = 49 > 0$, so that 2 does not lie between the roots, t_1 and t_2. Hence the roots are either both less than -2 or both greater than 2.

If both of the roots, t_1 and t_2 are negative, then their sum $a/16$ is less than -4, so that $a < -64$ and $t_1 t_2 = (2a - 15)/64 < 0$. But this yields a contradiction, as the roots have the same sign. Hence, we must have $2 < t_1 < t_2$, say, so that the four roots x_1, x_2, x_3, x_4 of the given equation are positive. Suppose that $x_1 \leq x_2 \leq x_3 \leq x_4$ with x_1 and x_4 the solutions of $x + (1/x) = t_2$ and x_2 and x_3 the solutions of $x + (1/x) = t_1$. (Explain why this alignment of indices is correct.) Note that $x_1 x_4 = x_2 x_3 = 1$. Since the four roots are in geometric progression with common ration $(x_4/x_1)^{1/3} = x_1^{-2/3}$, we find that

$$t_2 = x_1 + \frac{1}{x_1} = \left(x_1^{1/3} + \frac{1}{x_1^{1/3}}\right)\left(\left(x_1^{1/3} + \frac{1}{x_1^{1/3}}\right)^2 - 3\right) = t_1(t_1^2 - 3)$$

so that

$$\frac{a}{16} = t_1 + t_2 = t_1(t_1^2 - 2) \ ,$$

whence

$$a = t_1(16t_1^2 - 32) = t_1(at_1 - 2a + 15 - 32) = at_1^2 - (2a + 17)t_1$$

so that,

$$\begin{aligned} 0 &= -16t_1^2 + a(t_1 - 2) + 15 \\ &= -16t_1^2 + 16t_1(t_1^2 - 2)(t_1 - 2) + 15 \\ &= 16t_1^4 - 32t_1^3 - 48t_1^2 + 64t_1 + 15 \\ &= (2t_1 - 5)(2t_1 + 3)(4t_1^2 - 4t_1 - 1) \ . \end{aligned}$$

Therefore, $t_2 = 5/2$ and so $a = 170$.

Indeed, when $a = 170$, we find that $0 = 16x^4 - 170x^3 + 357x^2 - 170x + 16 = (x - 8)(x - 2)(2x - 1)(8x - 1)$.

Solution 2. Let the roots by ur^3, ur, ur^{-1}, ur^{-3}, with $u > 0$. Since the product of the roots is 1, we must have that $u = 1$. From the relationship between the coefficients and the roots, we have that

$$r^3 + r + r^{-1} + r^{-3} = \frac{a}{16}$$

and

$$r^4 + r^2 + 2 + r^{-2} + r^{-4} = \frac{2a+17}{16} \ .$$

Let $s = r + r^{-1}$ so that $s^3 - 2s = a/16$ and $s^4 - 3s^2 + 2 = (2a+17)/(16) = 2(s^3 - 2s) + (17/16)$. Hence

$$\begin{aligned} 0 &= s^4 - 2s^3 - 3s^2 + 4s + (15/16) \\ &= (1/16)(4s^2 - 4s - 15)(4s^2 - 4s - 1) = (1/16)(2s+3)(2s-5)(4s^2 - 4s - 1) \ . \end{aligned}$$

Since s must be real and its absolute value is not less than 2, $s = 5/2$ and so r is equal to either 2 or $1/2$. Therefore

$$a = 16\left(8 + 2 + \frac{1}{2} + \frac{1}{8}\right) = 170 \ .$$

216. Let x be positive and let $0 < a \leq 1$. Prove that

$$(1 - x^a)(1 - x)^{-1} \leq (1 + x)^{a-1} \ .$$

Solution 1. If $x = 1$, the inequality degenerates, but the related inequality $(1 - x^a) \leq (1 + x)^{a-1}(1 - x)$ holds. If $x > 1$, then, with $y = 1/x$, the inequality is equivalent to $(1 - y^a)(1 - y)^{-1} \leq (1 + y)^{a-1}$. (Establish this.) Hence, it suffices to show that the inequality holds when $0 < x < 1$.

By the concavity of the function $(1+x)^{1-a}$ for $x > -1$, we have that $(1+x)^{1-a} \leq 1 + (1-a)x$. (Observe that the tangent to the curve $y = (1 + x)^{1-a}$ at $(0, 1)$ is $y = 1 + (1 - a)x$.) Therefore

$$\begin{aligned} (1 - x) - (1 + x)^{1-a}(1 - x^a) &\geq (1 - x) - [1 + (1 - a)x](1 - x^a) \\ &- x - (1 - a)x + x^a + (1 - a)x^{a+1} = x^a + (1 - a)x^{a+1} - (2 - a)x \ . \end{aligned}$$

By the Arithmetic-Geometric Means Inequality,

$$\begin{aligned} \frac{x^a + (1-a)x^{a+1}}{2-a} &\geq x^{a(2-a)^{-1}} x^{(a+1)(1-a)/(2-a)} \\ &= x^{-(1-a)^2/(2-a)} x > x \ , \end{aligned}$$

since $x < 1$. The result now follows.

Solution 2. [A. Feiz Mohammadi] As above, we can restrict to the situation that $0 < x < 1$. Let $f(a) = (1 - x^a)(1 + x)^{1-a}$. Suppose, to begin with, we take $0 < a = m/n < 1$ for some positive integers m and n. Since $m < n$, $m - k < n - k < n$ for $0 < k$. Hence

$$\begin{aligned} x^n \left(\frac{1 - x^m}{1 - x}\right) &= \sum_{k=0}^{m-1} x^n x^k \leq \sum_{k=0}^{m-1} x^{m-k} x^k \\ &\leq m x^m < n x^m \ , \end{aligned}$$

whence (using the binomial expansion),

$$\begin{aligned} (1 + x^n)^{1/n} &\leq 1 + \frac{x^n}{n} \\ &\leq 1 + \frac{x^m(1-x)}{1 - x^m} = \frac{1 - x^{m+1}}{1 - x^m} \ . \end{aligned}$$

This inequality holds if we replace x by $x^{1/n}$. Therefore

$$(1 + x)^{1/n} \leq \frac{1 - x^{(m+1)/n}}{1 - x^{m/n}}$$

$$\Longrightarrow (1 - x^{m/n})(1+x)^{1-(m/n)} \le (1 - x^{(m+1)/n})(1+x)^{1-(m+1)/n} \ .$$

Thus, $f(m/n) \le f((m+1)/n)$.

Let u and v be two rationals with $0 < u < v < 1$, and let n be a common denominator, so that $u = m/n$ and $v = (m+p)/n$ for some positive integers m and p. Then $f(u) \le f(v)$. Let r_0 be a given rational in $(0,1)$, and let $\{r_k : k \ge 0\}$ be an increasing sequence for which $\lim_{k\to\infty} r_k = 1$. Since f is an increasing function of rational a,

$$1 - x = f(1) = \lim_{k\to\infty} f(r_k) \ge f(r_0) \ .$$

Suppose that a is any real with $0 < a < 1$. Suppose, if possible, that $f(a) > 1 - x$ and let $\epsilon = f(a) - (1-x) > 0$. Since f is continuous at a, there is a positive number δ with $0 < \delta < \min\,(a, 1-a)$ for which $|f(r) - f(a)| < \epsilon$ whenever $0 < |a - r| < \delta$. Let r be a rational satisfying this condition. Then $0 < r < 1$, $f(r) < f(1)$ and so

$$\epsilon > f(a) - f(r) = (f(a) - f(1)) + (f(1) - f(a)) > f(a) - f(1)$$

yielding a contradiction. The result follows.

Solution 3. [R. Furmaniak] Fix $x > 0$, $x \ne 1$ and let

$$F(a) = (1 - x^a)(1+x)^{1-a}(1-x)^{-1}$$

for $a > 0$. Note that $F(a) > 0$. Observe that, by the Arithmetic-Geometric Means Inequality,

$$2x^{(a+b)/2} \le x^a + x^b$$

so that

$$(1 - x^a)(1 - x^b) \le (1 - x^{(a+b)/2})^2 \ .$$

Hence

$$\sqrt{F(a)F(b)} \le F\left(\frac{a+b}{2}\right)$$

for $a, b > 0$, so that $\log F(a)$ is a concave function on the half-line $(0, \infty)$.

Now $F(1) = 1$ and $F(2) = (1+x)(1+x)^{-1} = 1$, so that $\log F(a)$ vanishes at $a = 1$ and $a = 2$. Hence, by the concavity,

$$F(a) \le 1 \Longleftrightarrow \log F(a) \le 0 \Longleftrightarrow 0 < a \le 1 \quad \text{or} \quad 2 \le a$$

and the result follows.

217. Let the three side lengths of a scalene triangle be given. There are two possible ways of orienting the triangle with these side lengths, one obtainable from the other by turning the triangle over, or by reflecting in a mirror. Prove that it is possible to slice the triangle in one of its orientations into finitely many pieces that can be rearranged using rotations and translations in the plane (but not reflections and rotations out of the plane) to form the other.

Solution 1. There are several ways of doing this problem. Observe that, if a geometric figure has a reflective axis of symmetry, then a rotation of 180° about a point on the axis (combined with a translation) will allow it to be superimposed upon its image reflected in an axis perpendicular to the reflective axis. For example, this applies to kites and isosceles triangles. So one strategy is to cut the triangle into finitely many pieces that have such a reflective axis of symmetry.

(a) Cut from the three vertices into the circumcentre of the triangle to obtain three isosceles triangles, which can be rearranged to give the other orientation.

(b) The triangle has at least one internal altitude. Cutting along this altitude yields two right triangles, each of which can be sliced along its median to the hypotenuse to give two isosceles triangles.

(c) Slice along the lines from the incentre of the triangle to the feet of the perpendiculars to the sides from the incentre. This yields three kites that can be moved to give the other orientation.

Solution 2. Superimpose the triangle onto its image obtained by reflecting in a line parallel to its longest side so that the corresponding side of one triangle contains the opposite vertex to this side of the other. Make cuts to produce the quadrilateral common to the triangle and its image. The remaining (isosceles) pieces of the triangle can be rotated to cover the corresponding parts of the image.

218. Let ABC be a triangle. Suppose that D is a point on BA produced and E a point on the side BC, and that DE intersects the side AC at F. Let $BE + EF = BA + AF$. Prove that $BC + CF = BD + DF$.

Solution 1. [O. Bormashenko] Produce CA to W so that $AW = AB$; produce FE to X so that $EX = EB$; produce FC to Y so that $CY = CB$; produce FD to Z so that $DZ = BD$. Then $\angle EXB = \angle EBX = \frac{1}{2}\angle FEB$ (exterior angle), and

$$FW = FA + AW = FA + AB = BE + EF = XE + EF = XF$$

so that $\angle FWX = \angle FXW = \frac{1}{2}\angle CFE$.

$\angle CBY = \angle CYB = \frac{1}{2}\angle BCF \Longrightarrow$

$$\angle XBY = \angle XBE - \angle CBY = \frac{1}{2}(\angle FEB - \angle BCF) = \frac{1}{2}\angle CFE$$

(exterior angle). Hence, $\angle XBY = \angle FWX = \angle YWX$ and $WBXY$ is concyclic.

Also,

$$\begin{aligned}\angle ZBW =& \angle ABW - \angle ABZ = \angle ABW - \angle DBZ = \frac{1}{2}(\angle CAB - \angle XDB) \\ =& \frac{1}{2}\angle DFA = \frac{1}{2}\angle CFE = \angle FXW = \angle ZXW\end{aligned}$$

and so $WBXZ$ is concyclic. Therefore, $WXYZ$ is concyclic and $\angle FZW = \angle XZW = \angle XYW = \angle XYF$.

Consider triangles ZFW and YFX. Since $FW = FX$, $\angle ZFW = \angle YFX$ and $\angle FZW = \angle XYZ$, $\Delta ZFW \equiv \Delta YFX$, and so $FZ = FY$. Therefore,

$$BC + CF = YC + CF = YF = ZF = ZD + DF = BD + DF\ .$$

Solution 2. [A. Feiz Mohammadi] Let $\angle EBF = u_1$, $\angle ABF = u_2$, $\angle BFE = v_1$ and $\angle BFA = v_2$. From the law of sines, we have that

$$EB : EF : BF = \sin v_1 : \sin u_1 : \sin(u_1 + v_1)$$

whence

$$(EB + EF) : BF = (\sin v_1 + \sin u_1) : \sin(u_1 + v_1)\ .$$

Similarly,

$$(AB + FA) : BF = (\sin v_2 + \sin u_2) : \sin(u_2 + v_2)\ .$$

Hence

$$\begin{aligned}
\frac{\sin u_1 + \sin v_1}{\sin(u_1+v_1)} = \frac{\sin u_2 + \sin v_2}{\sin(u_2+v_2)} &\Leftrightarrow \frac{\cos\frac{1}{2}(u_1-v_1)}{\cos\frac{1}{2}(u_1+v_1)} = \frac{\cos\frac{1}{2}(u_2-v_2)}{\cos\frac{1}{2}(u_2+v_2)} \\
&\Leftrightarrow \cos\frac{1}{2}(u_1-v_1)\cos\frac{1}{2}(u_2+v_2) = \cos\frac{1}{2}(u_2-v_2)\cos\frac{1}{2}(u_1+v_1) \\
&\Leftrightarrow \cos\frac{1}{2}(u_1+u_2+v_2-v_1) + \cos\frac{1}{2}(v_1+v_2+u_2-u_1) \\
&\qquad\qquad = \cos\frac{1}{2}(u_1+u_2+v_1-v_2) + \cos\frac{1}{2}(v_1+v_2+u_1-u_2) \\
&\Leftrightarrow \cos\frac{1}{2}(u_1+u_2+v_2-v_1) - \cos\frac{1}{2}(v_1+v_2+u_1-u_2) \\
&\qquad\qquad = \cos\frac{1}{2}(u_1+u_2+v_1-v_2) - \cos\frac{1}{2}(v_1+v_2+u_2-u_1) \\
&\Leftrightarrow \sin\frac{1}{2}(u_1+v_2)\sin\frac{1}{2}(v_1-u_2) = \sin\frac{1}{2}(u_2+v_1)\sin\frac{1}{2}(v_2-u_1) \\
&\Leftrightarrow \frac{\sin\frac{1}{2}(v_2+u_1)\cos\frac{1}{2}(v_2-u_1)}{\sin\frac{1}{2}(v_2-u_1)\cos\frac{1}{2}(v_2-u_1)} = \frac{\sin\frac{1}{2}(v_1+u_2)\cos\frac{1}{2}(v_1-u_2)}{\sin\frac{1}{2}(v_1-u_2)\cos\frac{1}{2}(v_1-u_2)} \\
&\Leftrightarrow \frac{\sin u_1 + \sin v_2}{\sin(v_2-u_1)} = \frac{\sin u_2 + \sin v_1}{\sin(v_1-u_2)} \\
&\Leftrightarrow \frac{\sin\angle FBC + \sin\angle BFC}{\sin\angle FCB} = \frac{\sin\angle FBD + \sin\angle DFB}{\sin\angle FDB} \\
&\Leftrightarrow \frac{FC+BC}{BF} = \frac{DF+DB}{BF} \Leftrightarrow BC+CF = BD+DF\ .
\end{aligned}$$

219. There are two definitions of an ellipse.

(1) An ellipse is the locus of points P such that the sum of its distances from two fixed points F_1 and F_2 (called *foci*) is constant.

(2) An ellipse is the locus of points P such that, for some real number e (called the *eccentricity*) with $0 < e < 1$, the distance from P to a fixed point F (called a *focus*) is equal to e times its perpendicular distance to a fixed straight line (called the *directrix*).

Prove that the two definitions are compatible.

Solution 1. Consider the following set of equivalent equations:

$$\begin{aligned}
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\Leftrightarrow \sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\Leftrightarrow x^2+2xc+c^2+y^2 &= 4a^2+x^2-2xc+c^2+y^2-4a\sqrt{(x-c)^2+y^2} \\
\Leftrightarrow \sqrt{(x-c)^2+y^2} &= a - \frac{xc}{a} = e\left(\frac{a}{e}-x\right)
\end{aligned}$$

where $e = c/a$. In applying the first definition, we may take the foci to be at the points $(c,0)$ and $(-c,0)$ and the sum of the focal radii to be $2a$. The final equation in the set describes the locus of a point whose distance from the focus $(c,0)$ is equal to e times the distance to the line $x = a/e$.

However, in applying the second definition, we can without loss of generality assume that the focus is at $(c,0)$ and the directrix is given by $x = d$. Where e is the eccentricity, let $a = de$. Then, reading up the equations, note that in going from the third to the second, both sides of the second have the same sign. Then the first equation describes a locus determined by the two foci condition.

Solution 2. In this solution, we start with the standard form of the equation for each definition and show that it describes the other locus.

In applying the first definition, place the foci at the points $(c, 0)$ and $(-c, 0)$, where $c > 0$, and let the ellipse be the locus of points P for which the sum of the distances to the foci is the constant value $2a > 0$. Thus, the equation of the locus is

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

$$\Leftrightarrow \sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

$$\Rightarrow x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + x^2 + 2cx + c^2 + y^2$$

$$\Leftrightarrow a\sqrt{(x+c)^2+y^2} = a^2 + cx$$

$$\Rightarrow a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$\Leftrightarrow (a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2) \ .$$

Let $b^2 = a^2 - c^2$. Then the equation can be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ .$$

This equation can also be written

$$y^2 = b^2 - \frac{b^2x^2}{a^2} \ .$$

Consider the line $x = d$, where $d > 0$ and let P be a point on the ellipse, F be the focus at $(c, 0)$ and Q be the foot of the perpendicular from P to the line $x = d$. We want to select d so that the ratio $PF^2 : PQ^2$ is independent of $P(x, y)$. Now

$$\begin{aligned}
\frac{PF^2}{PQ^2} &= \frac{(x-c)^2+y^2}{(d-x)^2} \\
&= \frac{x^2-2cx+c^2+b^2-(b^2/a^2)x^2}{x^2-2dx+d^2} \\
&= \frac{(a^2-b^2)x^2-2a^2cx+(b^2+c^2)a^2}{a^2x^2-2a^2dx+a^2d^2} \\
&= \frac{c^2}{a^2}\left[\frac{x^2-(2a^2/c)x+(a^4/c^2)}{x^2-2dx+d^2}\right] \ .
\end{aligned}$$

The quantity in the square brackets is equal to 1 when $d = a^2/c$. Thus, when $d = a^2/c$, $PF^2 : PQ^2 = c^2 : a^2$, a constant ration. Define $e = c/a$. Note that $e < 1$. Then we find that $PF = ePQ$ and $a = de$.

On the other hand, start with the focus-directrix definition of an ellipse with eccentricity e, focus at $(0, 0)$ and directrix $x = d$. Then

$$x^2 + y^2 = e^2(x^2 - 2dx + d^2) \Leftrightarrow (1-e^2)\left[x + \frac{de^2}{1-e^2}\right]^2 + y^2 = d^2e^2 + \frac{d^2e^4}{1-e^2} = \frac{d^2e^2}{1-e^2}$$

$$\Leftrightarrow \left[x + \frac{de^2}{1-e^2}\right]^2 + \frac{y^2}{1-e^2} = \left(\frac{de}{1-e^2}\right)^2 \ .$$

Setting $y = 0$, we can check that the curve cuts the $x-$axis at the points $((de)/(1+e), 0)$ and $((-de)/(1-e), 0)$. Define a to be equal to

$$\frac{1}{2}\left(\frac{de}{1+e} + \frac{de}{1-e}\right) = \frac{de}{1-e^2} \ ,$$

$c = ea$ and $b = \sqrt{a^2 - c^2}$. Then the equation of the focus-directrix locus becomes

$$(x + c)^2 + \frac{y^2}{1 - (c^2/a^2)} = a^2$$

$$\Leftrightarrow \frac{(x + c)^2}{a^2} + \frac{y^2}{b^2} = 1 \ ,$$

which is a shift of the locus of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

c units to the left.

Since it is not completely clear that the latter form indeed represents the locus according to the two-foci definition, we show that the sum of the distances from any point on the curve to the points $(0, 0)$ and $(-2c, 0)$ is constant. Note that $y^2 = (b^2/a^2)[b^2 - x^2 - 2cx]$, from which

$$x^2 + y^2 = \frac{(cx - b^2)^2}{a^2}$$

and

$$\begin{aligned}(x + 2c)^2 + y^2 &= x^2 + 4cx + 4c^2 + (b^2/a^2)[b^2 - x^2 - 2cx] \\ &= (1/a^2)[(a^2 - b^2)x^2 + 2c(2a^2 - b^2)x + 4a^2c^2 + (a^2 - c^2)^2] \\ &= (1/a^2)[c^2x^2 + 2c(a^2 + c^2) + (a^2 + c^2)^2] \\ &= \frac{(cx + a^2 + c^2)^2}{a^2} \ .\end{aligned}$$

We need to ensure which square root is correct when we calculate the sum of the distances. Note that

$$\frac{b^2}{c} = \frac{a^2}{c} - c = \frac{a}{e} - c = \frac{d}{1 - e^2} - \frac{de^2}{1 - e^2} = d$$

so that $x < d = b^2/c$. Note also that

$$\frac{a^2 + c^2}{c} - \frac{de}{1 - e} = \frac{a^2}{c} + c - \frac{de}{1 - e} = \frac{d}{1 - e^2}[1 + e^2 - e(1 + e)] = \frac{d}{1 + e} > 0 \ .$$

Hence

$$-\frac{a^2 + c^2}{c} < -\frac{de}{1 - e} \le x$$

at all points on the curve. Hence

$$\begin{aligned}\sqrt{x^2 + y^2} + \sqrt{(x + 2c)^2 + y^2} &= \frac{1}{a}[(b^2 - cx) + (cx + a^2 + c^2)] \\ &= \frac{a^2 + b^2 + c^2}{a} = \frac{2a^2}{a} = 2a \ ,\end{aligned}$$

a constant.