Solutions

Notes. A *partition* of the positive integer n is a representation (up to order) of n as a sum of not necessarily distinct positive integers, *i.e.*, $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. The number of distinct partitions is denoted by $p(n)$. Thus, $p(6) = 11$ since 6 can be written as $6 = 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 3+1+1+1 = 2+2+2 = 2+2+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1$.

Sources. 234. *Bulgarian math competitions - selected problems*, Tonov I. *et al*, Regalia-6, Sofia, 2001. 235, 236. Junior Balkan Math Olympiad, 2002. 237. Balkan Math Olympiad, 2002. 238, 239. *Mathematics Plus*, issues 3, 4, 2002, Sofia, 240. National Mathematical Olympiad, 1999, Bulgaria, Regional Round.

234. A square of side length 100 is divided into 10000 smaller unit squares. Two squares sharing a common side are called *neighbours*.

(a) Is it possible to colour an even number of squares so that each coloured square has an even number of coloured neighbours?

(b) Is it possible to colour an odd number of squares so that each coloured square has an odd number of coloured neighbours?

Solution. [Y. Zhao] (a) Yes, it is possible in many ways to perform the task. For example, colour any two nonadjacent squares, and both of them will have zero coloured neighbours. So there are evenly many (2) coloured squares, each with an even number (0) of coloured neighbours.

(b) Suppose, if possible, we could colour an odd number of squares so that each has an odd number of coloured neighbours. Let us count the number of segments or edges that connect two coloured neighbours. Since for each coloured square there is an odd number of coloured neighbours, then the total number of their common sides is the sum of an odd number of odd terms, and so is odd. However, two coloured neighbours share each of these common edges, therefore each coloured neighbour is counted twice in the sum; thus, the sum should be even. This is a contradiction. So, it is impossible to colour an odd number of squares so that each has an odd number of coloured neighbours.

235. Find all positive integers, N, for which:

(i) N has exactly sixteen positive divisors: $1 = d_1 < d_2 < \cdots < d_{16} = N$;

(ii) the divisor with the *index* d_5 (namely, d_{d_5}) is equal to $(d_2 + d_4) \times d_6$ (the product of the two).

Solution. There are some preliminary easy observations:

(1) Since N has exactly sixteen positive divisors and d_5 is an index, $d_5 \leq 16$. On the other hand, d_6 is a proper divisor of d_{d_5}, so $d_6 \leq d_{d_5}$. Thus $6 < d_5 \leq 16$.

(2) If N were odd, all its factors would be odd. But, by (ii), the factor d_{d_5} would be the product of an even and an odd number, and so be even. But this would given N an even divisor and lead to a contradiction.

(3) Recall that, if $N = \prod p_i^{k_i}$ is the prime factor decomposition, then the number of all divisors, including 1 and N is $\prod(1 + k_i)$. [To understand this formula, think how we can form any of the divisors of N; we have to choose its prime factors, each to any of the possible exponents. For an arbitrary prime factor p_i there are $(1 + k_i)$ possibility for the exponent (from 0 to k_i inclusive). In particular, the factor 1 corresponds to taking all exponents 0, and N to taking all exponents to be the maximum k_i.] It can be checked that there are five cases for the prime factorization of N; (i) $N = p^{15}$, $N = p_1^7 p_2$; (iii) $N = p_1^3 p_2 p_3$; (iv) $N = p_1^3 p_2^3$; (v) $N = p_1 p_2 p_3 p_4$.

We now put all of this together, and follow the solution of K.-C. R. Tseng. From (1), $d_2 = 2$.

If d_4 is composite (*i.e.* not a square), then $d_4 = 2d_3$ is even. Since $d_2 + d_4$ divides a factor d_{d_5} of N, it divides N. Since $d_2 + d_4 = 2(1 + d_3)$, $1 + d_3$ divides N. But then $1 + d_3$ would equal $d_4 = 2d_3$, which is impossible. If d_4 were a perfect square, then it must equal either 4 or 9 (since $d_4 < d_5 \leq 16$). In either case, $d_3 = 3$, and 6 must be one of the factors. This excludes the possibility that $d_4 = 9$, since 6 should preceded

9 in the list of divisors. On the other hand, if $d_4 = 4$, then d_5 must be equal to either 5 or 6, which is not possible by (1).

Hence, d_4 must be a prime number, and so one of 3, 5, 7, 11, 13. Since $d_3 \geq 3$, $d_4 \neq 3$.

Suppose that $d_4 = 5$. Then $d_2 + d_4 = 7$ must divide N. Thus d_5 or d_6 must be 7. If $d_5 = 7$, then $d_3 \neq 3$, for otherwise 6 would be a factor between d_4 and d_5. But then $d_3 = 4$, so that $N = 2^2 \cdot 5 \cdot 7 \cdot K$ where K is a natural number. But N must have 16 divisors, and the only way to obtain this is to have 2^3 rather than 2^2 in the factorization. Thus, $d_6 = 8$ and $d_7 = 10$. But then $d_{d_5} = d_7 \neq (d_2 + d_4)d_6$. So $d_5 = 7$ is rejected and we must have $d_6 = 7$. This entails that $d_5 = 6$. But this denies the equality of $d_6 = d_{d_5}$ and $(d_2 + d_4)d_6$. We conclude that $d_4 \neq 5$.

Suppose that $d_4 = 7$. Then $d_2 + d_4 = 9$ is a factor of N, so $d_3 = 3$. Then 6 must be a factor of N; but there is not room for 6, and this case is impossible.

Suppose that $d_4 = 11$. Then $d_2 + d_4 = 13$ divides N, and is either d_5 (when 12 is not a factor) or d_6 (when 12 is a factor). If $d_5 = 13$, then d_3 is either a prime number less than 11 or 4. It cannot be 3, as there is no room to fit the divisor 6. If $d_3 = 4$, then $N = 2^2 \cdot 11 \cdot 13 \cdot K$ and the only way to get 16 divisors is for the exponent of 2 to be 3. Thus, 8 divides N, but there is no room for this divisor. Similarly, if $d_5 = 5$, there is no room for 10.

Finally (with $d_4 = 11, d_5 = 13$), if $d_3 = 7$, we already have four prime divisor of N, and this forces $N = 2 \cdot 7 \cdot 11 \cdot 13 = 2002$. We have that the divisors in increasing order are 1, 2, 7, 11, 13, 14, 22, 26, 77, 91, 143, 154, 182, 286, 1001, 2002, and all the conditions are satified.

When $d_4 = 11$, $d_6 = 13$, then $d_5 = 12$, so that 3, 4, 6 are all factors of N; but there is no room for them between d_2 and d_4.

The remaining case is that $d_4 = 13$, which makes $d_2 + d_4 = 15$ a factor of N; but there is no room for both 3 and 5 between d_2 and d_4. We conclude that $N = 2002$ is the only possibility.

236. For any positive real numbers a, b, c, prove that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2} \ .$$

Solution. [G.N. Tai] Apply the AM-GM Inequality to get

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq 3\sqrt[3]{\frac{1}{abc(a+b)(b+c)(c+a)}}$$

$$a + b + c \geq 3\sqrt[3]{abc}$$

$$a + b + c = \frac{1}{2}((a+b) + (b+c) + (c+a)) \geq \frac{3}{2}\sqrt[3]{(a+b)(b+c)(c+a)} \ .$$

Multiplying these inequalities together and dividing by $(a+b+c)^2$ yields the result. Equality occurs if and only if $a = b = c$.

237. The sequence $\{a_n : n = 1, 2, \cdots\}$ is defined by the recursion

$$a_1 = 20 \qquad\qquad a_2 = 30$$

$$a_{n+2} = 3a_{n+1} - a_n \qquad\qquad \text{for} \ \ n \geq 1 \ .$$

Find all natural numbers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

Solution. [R. Marinov] The first few terms of the sequence are 20, 30, 70, 180, 470, 1230. Observe that

$$0 = (a_{n+1} - a_{n-1})(a_{n+1} + a_{n-1} - 3a_n) \Leftrightarrow a_{n+1}^2 - 3a_{n+1}a_n = a_{n-1}^2 - 3a_n a_{n-1}$$

so that

$$a_{n+1}^2 - 3a_n a_{n+1} + a_n^2 = a_n^2 - 3a_{n-1}a_n + a_{n-1}^2$$

for $n \geq 2$. Hence $a_{n+1}^2 - 3a_{n+1}a_n + a_n^2$ is a constant for $N \geq 2$, and its value is $30^2 - 2 \cdot 30 \cdot 20 + 20^2 = -500$.

Now, $1 + 5a_n a_{n+1} = 501 - 500 + 5a_n a_{n+1} = 501 + (a_{n+1} + a_n)^2$ for each $n \geq 1$. Since $1 + 5a_n a_{n+1} = k^2$ is equivalent to

$$3 \times 167 = 501 = (k - (a_{n+1} + a_n))(k + (a_{n+1} - a_n) \ ,$$

we must have that either (i) $A - (a_{n+1} + a_n) = 1$ and $A + (a_{n+1} + a_n) = 501$ or (ii) $A - (a_{n+1} + a_n) = 3$ and $A + (a_{n+1} + a_n) = 167$. The second possibility leads to $a_{n+1} + a_n = 82$ which is not divisible by 10 and so cannot occur. The first possibility leads to $a_{n+1} + a_n = 250$, which occurs when $n = 3$. Since the sequence is increasing (prove this!), this is the only possibility.

238. Let ABC be an acute-angled triangle, and let M be a point on the side AC and N a point on the side BC. The circumcircles of triangles CAN and BCM intersect at the two points C and D. Prove that the line CD passes through the circumcentre of triangle ABC if and only if the right bisector of AB passes through the midpoint of MN.

Note: Figures 1, 2, 3 accompany this solution.

Solution. Denote the circumcentres of the triangles ABC, ANC and BMN by O, O_1 and O_2 respectively. Denote also their circumcircles by K, K_1 and K_2 respectively, and the radii of these circles by R, R_1 and R_2 respectively. (See figure 1.) The common chord CD of K_1 and K_2 is perpendicular to O_1O_2. Thus, $O \in CD \Longleftrightarrow CO \perp O_1O_2$.

We prove two lemmata.

`Lemma 1.` Let M_1 be the perpendicular projection of the point M onto AB and N_1 the projection of the point N onto AB. The right bisector of AB, the line S_{AB}, passes through the midpoint of MN if and only if $AN_1 = BM_1$. (See figure 2.)

`Proof.` Note that MM_1N_1N is a trapezoid with bases parallel to S_{AB}. Recall that the midline of a trapezoid has the following property: *the segment that connects the midpoints of the two nonparallel sides is parallel to the bases and its length is the average of the lengths of the two parallel sides.* As a direct consequence, *a line passing through one of the midpoints of the two nonparallel sides and is parallel to the bases must pass through the midpoint of the other side.* Applying this yields that S_{AB} passes through the midpoint of MN if and only if S_{AB} passes through the midpoint of M_1N_1. Since S_{AB} intersects AB at its midpoint, this is equivalent to S_{AB} passes through the midpoint of $M_1N_1 \Leftrightarrow AB$ and M_1N_1 have the same midpoint, which is equivalent to $AM_1 = BN_1$ or $AN_1 = BM_1$ ♠.

`Lemma 2.` The diagonals d_1 and d_2 of the quadrilateral $PQRS$ are perpendicular if and only if its sides a, b, c, d satisfy the relationship $a^2 + c^2 = b^2 + d^2$. ((a, c) and (b, d) are pairs of opposite sides.)

`Proof.` (To follow the steps of the proof, please draw an arbitrary convex quadrilateral $PQRS$ with the respective lengths of SR, RQ, QP and PS given by a, b, c and d.) Let d_1 and d_2 intersect at I, and let

$$\angle PIQ = \theta \ , \quad |IP| = t \ , \quad |IQ| = z \ , \quad |IR| = y \ , \quad |IS| = x \ .$$

The Law of Cosines applied to triangles PQI, QRI, RSI and SPI yields

$$a^2 = x^2 + y^2 - 2xy\cos\theta$$

$$c^2 = z^2 + t^2 - 2zt\cos\theta$$

$$b^2 = y^2 + z^2 + 2yz\cos\theta$$

$$d^2 = x^2 + t^2 + 2xt\cos\theta \ .$$

As $a^2 + c^2 = b^2 + d^2$ is equivalent to $(xy + zt + yz + xt)\cos\theta = 0$, or $\cos\theta = 0$, the result follows. ♠

Let us return to the problem. Consider (in figure 1) the quadrilateral CO_1OO_2. We already know from the foregoing that

- CD passes through $O \Leftrightarrow CO \perp O_1O_2$;
- $CO \perp O_1O_2 \Leftrightarrow O_1C^2 + OO_2^2 = O_2C^2 + OO_1^2$;
- $AN_1 = BM_1 \Leftrightarrow S_{AB}$ passes through the midpoint of MN.

So to complete the solution, it is necessary to prove that

$$O_1C^2 + OO_2^2 = O_2C^2 + OO_1^2 \Longleftrightarrow AN_1 = BM_1 \ .$$

From the Law of Cosines,

$$OO_1^2 = O_1C^2 + OC^2 - 2O_1C \cdot OC \cdot \cos \angle O_1CO$$

and

$$OO_2^2 = O_2C^2 + OC^2 - 2O_2C \cdot OC \cdot \cos \angle O_2CO$$

from which

$$O_1C^2 + OO_2^2 = O_2C^2 + OO_1^2 - 2OC \cdot (O_2C \cos \angle O_2CO) - O_1C \cos \angle O_1CO) \ .$$

We need to establish that (i) $\angle O_1CO = \angle NAB$ and (ii) $\angle O_2CO = \angle MBA$. (See figure 3.) Ad (i), $\angle AO_1N = 2\angle ACN = 2\alpha$ and $\angle CO_1N = 2\angle CAN = 2\beta$, say, so that $\angle CO_1A = 2(\alpha+\beta)$. The common chord CA of K_1 and K is right bisected by O_1O, so that $\angle CO_1A = 2\angle CO_1O$ and $\angle CO_1O = \alpha + \beta$. On the other hand, $\angle COO_1 = \frac{1}{2}\angle COA = \angle CBA = \gamma$, say. Hence, $\angle O_1CO = 180° - (\alpha + \beta + \gamma)$. Also, $\angle ANB = \alpha + \beta$ and $\angle NAB = 180° - (\alpha + \beta + \gamma) = \angle O_1CO$. Similarly, (ii) can be shown.

From the extended Law of Sines involving the circumradius, we have that $2R_1 = AN/\sin C$ and $2R_2 = MB/\sin C$. It follows that

$$O_2C \cos \angle O_2CO - O_1C \cos \angle O_1CO = 0$$

$$\Leftrightarrow R_2 \cdot \cos \angle MBA - R_1 \cdot \cos \angle NAB = 0$$

$$\Leftrightarrow MB \cos \angle MBA = AN \cos \angle NAB \ .$$

However, $MB \cos \angle MBA = BM_1$ and $AN \cos \angle NAB = AN_1$ (the lengths of the projections on AB). The result now follows, that CD passes through O if and only if S_{AB} passes through the midpoint of MN.

239. Find all natural numbers n for which the diophantine equation

$$(x + y + z)^2 = nxyz$$

has positive integer solutions x, y, z.

Solution. Let $(n; x, y, z) = (n; u, v, w)$ be a solution of the equation. Then the quadratic equation

$$t^2 + (2u + 2v - nuv)t + (u + v)^2 = 0$$

has two solutions, w and a second one w' for which $ww' = (u + v)^2 > 0$ (product of the roots). Since $w + w' = -(2u + 2v - nuv)$, an integer, w' must be a positive integer, and so $(n; x, y, z) = (n; u, v, w')$ is a solution of the equation. If $w > (u+v)$, then $w' < (u+v)$. It follows that, if there is a solution, we can repeat the process long enough using any two of the three variables as fixed to always find solutions $(n; x, y, z)$ of the equation for which $z \le x + y$, $y \le x + z$ and $x \le x + y$. So we impose this additional restriction in our search. Wolog, we can also suppose that $1 \le x \le y \le z$.

Suppose $x = 1$. Since $z \le x+y = 1+y$, $(x, y, z) = (1, r, r)$ or $(1, r, r+1)$. The first leads to $(2r+1)^2 = nr^2$ or $1 = r(nr - 4r - 4)$, whence $(n; r) = (9, 1)$. The second leads to $4(r + 1)^2 = nr(r + 1)$, or $4 = (n - 4)r$; this yields $(n; r) = (8; 1), (6; 2), (5; 4)$. Thus, the four solutions with $x = 1$ are

$$(n; x, y, z) = (5; 1, 4, 5), (6; 1, 2, 3); (8; 1, 1, 2); (9; 1, 1, 1) \ .$$

Suppose $x \ge 2$. Then

$$nxyz = (x + y + z)(x + y + z) \le (z + z + z)(x + y + x + y) = 6z(x + y)$$

so that $nxy \le 6(x + y)$. Rearranging the terms and adding 36 to both sides yields

$$(nx - 6)(ny - 6) \le 36 \ .$$

Since $2 \le x \le y$, we find that $(2n - 6)(2n - 6) \le 36$ so that $0 \le n \le 6$. Checking turns up the additional solutions

$$(n; x, y, z) = (1; 9, 9, 9), (2; 4, 4, 8); (3; 3, 3, 3); (4; 2, 2, 4) \ .$$

Thus, the only natural numbers n for which a solution exists are 1, 2, 3, 4, 5, 6, 8, 9.

240. In a competition, 8 judges rate each contestant "yes" or "no". After the competition, it turned out, that for any two contestants, two judges marked the first one by "yes" and the second one also by "yes"; two judges have marked the first one by "yes" and the second one by "no"; two judges have marked the first one by "no" and the second one by "yes"; and, finally, two judges have marked the first one by "no" and the second one by "no". What is the greatest number of contestants?

Solution. Let n be the number of contestants. Then, the marks of the judges for each of them can be recorded in a column of eight zeros or ones, as follows: there is a 1 on the ith position of the number if the ith judge has marked this contestant by "yes" and there is a 0 in this position if the ith judge has marked this contestant by "no". This way, the information about the marks of the contestants will be recorded in an $n \times 8$ table. Now, the given condition implies that the 2×8 table formed by any two columns of the above table has exactly two rows of each of 00, 01, 10, 11. Denote this property by $(*)$. We will now show that eight columns with any pair having this property do not exist.

Suppose the contrary, and consider a table with eight columns. Interchanging 1 and 0 in any column does not change the property $(*)$, so, wolog, we can assume that the first row consists solely of 0s. Let there be a_i 0s in the ith row. Then $\sum_{i=1}^{8} a_i = 8 \times 4 = 32$ and $\sum_{i=2}^{8} a_i = 32 - 8 = 24$. Next, we will count the number of pairs of two 0s that can appear in the lines of the table in two different ways.

(i) In the ith row, there are a_i 0s. We can choose two of them in $\binom{a_i}{2}$ ways, so the number of possible pairs in all rows is $\sum_{i=1}^{8} \binom{a_i}{2}$.

(ii) There are 8 columns. We can choose two of them in $\binom{8}{2} = 28$ ways. In each selection, there are exactly two rows with 00, so that all the ways to get combinations of two 0s is $2 \times 28 = 56$. Thus,

$$\sum_{i=1}^{8} \binom{a_i}{2} = 56 \ .$$

We have that

$$\begin{aligned}\sum_{i=1}^{8} \binom{a_i}{2} &= \frac{a_1(a_1 - 1)}{2} + \sum_{i=2}^{8} \frac{a_i(a_i - 1)}{2} \\ &= 28 - \frac{1}{2}\sum_{i=2}^{8} a_i + \frac{1}{2}\sum_{i=2}^{8} a_i^2 = 28 - 12 + \frac{1}{2}\sum_{i=2}^{8} a_i^2 \ ,\end{aligned}$$

from which $\sum_{i=2}^{8} a_i^2 = 2(56-28+12) = 80$. From the ineqaulity of the root mean square and the arithmetic mean, we have that

$$\frac{a_2^2 + \cdots + a_8^2}{7} \geq \left(\frac{a_2 + \cdots + a_8}{7}\right)^2 = \frac{576}{49} .$$

whence $80 = \sum_{i=2}^{8} a_i^2 \geq 576/7 > 82$, which is false. Therefore, we must conclude that there cannot be eight columns with condition $(*)$. However, we can realize this condition with a table of seven columns:

0	0	0	0	0	0	0
0	1	1	1	1	0	0
0	1	1	0	0	1	1
0	0	0	1	1	1	1
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1

Thanks to Emil Kolev, Sofia, Bulgaria for this problem.