Solutions for the December Problems

192. Let ABC be a triangle, D be the midpoint of AB and E a point on the side AC for which $AE = 2EC$. Prove that BE bisects the segment CD.

In the following solutions, F is the intersection point of BE and CD.

Solution 1. Let G be the midpoint of AE. Then $AG = GE = EC$ and $DG \| BE$. In triangle ADC, $DG \| FE$ and $GE = EC$, from which it follows that $DF = FC$, as required.

Solution 2. Let $u = [ADF] = [BDF]$ (where $[\cdots]$ denotes area), $v = [AFE]$, $w = [CFE]$ and $z = [BFC]$. Then $2u + v = 2(w + z)$ and $v = 2w$, whence $2u = 2z$ and $u = z$. It follows from this (two triangles with the same height and equal collinear bases) that F is the midpoint of CD.

Solution 3. By Menelaus' Theorem, applied to triangle ACD and transversal BFE,

$$\frac{CE}{EA} \cdot \frac{AB}{BD} \cdot \frac{DF}{FC} = -1 ,$$

so that $\frac{1}{2} \cdot (-2) \cdot (DF/FC) = -1$ and $DF = FC$, as desired.

Solution 4. [T. Yue] Let K be the midpoint of AC; then $BC = 2DK$ and $BC \| DK$. Suppose that BE produced and DK produced meet at H. Since triangles EBC and EHK are similar and $EC = 2EK$, it follows that $BC = 2KH$ and so $DH = BC$. Thus, $DHCB$ is a parallelogram whose diagonals BH and CD must bisect each other. The result follows.

Solution 5. Place the triangle in the cartesian plane so that $B \sim (0, 0)$, $C \sim (3, 0)$ and $A \sim (6a, 6b)$. Then $D \sim (3a, 3b)$, $E \sim (2(a + 1), 2b)$ and the lines BE and and CD have the respective equations $y = bx/(a + 1)$ and $y = b(x - 3)/(a - 1)$. These lines intersect at the point $((3/2)(a + 1), (3/2)b)$, and the result follows.

Solution 6. [L. Chen] $[BDE] = [ADE] = \frac{1}{2}[ABE] = [BEC]$. Let M and N be the respective feet of the perpendiculars from D and C to BE. Then $[BDE] = [BEC] \Rightarrow DM = CN$. Since DMF and CNF are similar right triangles with $DM = CN$, they are congruent and so $DF = CF$.

Solution 7. [F. Chung; Y. Jean] As in the previous solution, $[BDE] = [BEC]$. Therefore,

$$DF : FC = [DEF] : [CEF] = [DBF] : [CBF] = ([DEF] + [DBF]) : ([CEF] + [CBF])$$
$$= [BDE] : [BEC] = 1 : 1 .$$

Solution 8. [Y. Wei] Let U be a point on BC such that $DU \| AC$. Suppose that DU and BE intersect in V. Then $2EC = AE = 2DV$, so that $DV = EC$. Also $\angle VDF = \angle ECF$ abd $\angle DFV = \angle ECF$, so that triangles DVF and CEF are congruent. Hence $DF = FC$.

Solution 9. Let AF produced meet BC at L. By Ceva's Theorem,

$$\frac{AD}{DB} \cdot \frac{BL}{LC} \cdot \frac{CE}{AE} = 1 ,$$

whence $BL = 2LC$ and, so, $LE \| AB$. Since the triangles ABC and ELC are similar with factor 3, $AB = 3EL$. Let EL intersect CD at M. Then the triangles AFB and LFE are similar, so that $FD = 3FM$. But,

$$FD + FM + MC = DC = 3MC \Rightarrow 2FM = MC \Rightarrow FC = FM + MC = 3FM = FD ,$$

as desired.

Solution 10. [H. Lee] Let $\mathbf{u} = \overrightarrow{DB}$, $\mathbf{v} = \overrightarrow{EC}$, $\mathbf{a} = \overrightarrow{BF}$, $\lambda\mathbf{a} = \overrightarrow{FE}$, $\mathbf{b} = \overrightarrow{CF}$ and $\mu\mathbf{b} = \overrightarrow{FD}$. Then

$$\mathbf{a} + \mu\mathbf{b} + \mathbf{u} = \mathbf{0}$$

and

$$\mathbf{b} + \lambda \mathbf{a} + \mathbf{v} = \mathbf{0} \ .$$

Hence

$$\mathbf{u} = -\mathbf{a} - \mu \mathbf{b} \quad \text{and} \quad \mathbf{v} = -\mathbf{b} - \lambda \mathbf{a} \ .$$

Therefore, from triangle ABE,

$$\begin{aligned} \mathbf{0} &= (\lambda + 1)\mathbf{a} - 2\mathbf{v} + 2\mathbf{u} \\ &= (\lambda + 1)\mathbf{a} + 2\mathbf{b} + 2\lambda\mathbf{a} - 2\mathbf{a} - 2\mu\mathbf{b} \\ &= (3\lambda - 1)\mathbf{a} + 2(1 - \mu)\mathbf{b} \ . \end{aligned}$$

Since $\{\mathbf{a}, \mathbf{b}\}$ is a linearly independent set, $\lambda = 1/3$ and $\mu = 1$, yielding the desired result.

Solution 11. [M. Zaharia] Place masses 1, 1, 2, respectively, at the vertices A, B, C. We locate the centre of gravity of these masses in two ways. Since the masses at A and B have their centre of gravity at D, we can get an equivalent system by replacing the masses at A and B by a mass 2 at the point D. The centre of gravity of the original set-up is equal to the centre of gravity of masses of 2 placed at each of D and C, namely at the midpoint of CD.

On the other hand, the centre of gravity of the masses at A and C is at E. So the centre of gravity of the original set-up is equal to the centre of gravity of a mass 3 located at E and a mass 1 located at B, namely on the segment BE (at the point F for which $BF = 3FE$). Since both BE and CD contain the centre of gravity of the original set-up, the result follows.

Solution 12. Place the triangle in the complex plane with C at 0, B at $12z$ and A at 12. Then D is located at $6(z+1)$ and E at 4. Let P be the midpoint $3(z+1)$ of CD. Then, BP and PE are collinear since

$$12z - 3(z + 1) = 3(3z - 1) = 3[3(z + 1) - 4] \ ,$$

i.e., the vector $\overrightarrow{BP}$ is a real multiple of $\overrightarrow{PE}$. The result follows.

193. Determine the volume of an isosceles tetrahedron for which the pairs of opposite edges have lengths a, b, c. Check your answer independently for a regular tetrahedron.

Solution 1. The edges of the tetrahedron can be realized as the diagonals of the six faces of a rectangular parallelepiped with edges of length u, v, w in such a way that $a^2 = v^2 + w^2$, $b^2 = u^2 + w^2$ and $c^2 = u^2 + v^2$. The tetrahedran can be obtained from the parallelepiped by trimming away four triangular pyramids each with three mutually perpendicular faces (surrounding a corner of the parallelepiped) and three pairwise orthogonal edges of lengths u, v, w. Hence the volume of the tetrahedron is equal to

$$uvw - 4((1/6)uvw) = (1/3)uvw \ .$$

¿From the foregoing equations, $2u^2 = b^2 + c^2 - a^2$, $2v^2 = c^2 + a^2 - b^2$ and $2w^2 = a^2 + b^2 - c^2$. (By laying out the tetrahedron flat, we see that the triangle of sides a, b, c is acute and the right sides of these equations are indeed positive.) It follows that the volume of the tetrahedron is

$$\frac{\sqrt{2}}{12}\sqrt{(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)} \ .$$

In the case of a regular tetrahedron of side 1, the height is equal to $\sqrt{2/3}$ and the area of a side is equal to $\sqrt{3}/4$, and the formula checks out.

Solution 2. [D. Yu] Let the base of the tetrahedron be triangle ABC, eith $a = |BC| = |AD|$, $b = |AC| = |BD|$, $c = |AB| = |CD|$; let P be the foot of the perpendicular from D to the plane of ABC and let $h = |DP|$. Then $|AP| = \sqrt{a^2 - h^2}$, $|BP| = \sqrt{b^2 - h^2}$, $|CP| = \sqrt{c^2 - h^2}$.

Suppose that $\alpha = \angle BCP$ and $\beta = \angle ACP$. Then using the Law of Cosines on triangles BCP, ACP and ABC, we obtain that

$$\cos\alpha = \frac{a^2 + c^2 - b^2}{2a\sqrt{c^2 - h^2}}$$

$$\cos\beta = \frac{b^2 + c^2 - a^2}{2b\sqrt{c^2 - h^2}}$$

and

$$\cos(\alpha + \beta) = \frac{a^2 + b^2 - c^2}{2ab} ,$$

whence

$$\frac{a^2 + b^2 - c^2}{2ab} =$$

$$\frac{(a^2 + c^2 - b^2)(b^2 + c^2 - a^2) - \sqrt{4a^2(c^2 - h^2) - (a^2 + c^2 - b^2)^2}\sqrt{4b^2(c^2 - h^2) - (b^2 + c^2 - a^2)^2}}{4ab(c^2 - h^2)} .$$

Shifting terms and squaring leads to

$$[2(a^2+b^2-c^2)(c^2-h^2)-(a^2+c^2-b^2)(b^2+c^2-a^2)]^2 = [4a^2(c^2-h^2)-(a^2+c^2-b^2)^2][4b^2(c^2-h^2)-(b^2+c^2-a^2)^2] .$$

With $u = b^2 + c^2 - a^2$, $v = c^2 + a^2 - b^2$, $w = a^2 + b^2 - c^2$, $z = c^2 - h^2$, this can be rendered

$$\begin{aligned} 0 &= [2wz - uv]^2 - [4a^2z - v^2][4b^2z - u^2] \\ &= z[4(w^2 - 4a^2b^2)z - 4(uvw - a^2u^2 - b^2v^2)] \end{aligned}$$

so that

$$c^2 - h^2 = z = \frac{a^2u^2 + b^2v^2 - uvw}{4a^2b^2 - w^2}$$

and

$$h^2 = \frac{4a^2b^2c^2 + uvw - a^2u^2 - b^2v^2 - c^2w^2}{4a^2b^2 - w^2} .$$

Now

$$\begin{aligned} 4a^2b^2 - w^2 &= -a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 \\ &= (a + b + c)(a + b - c)(b + c - a)(c + a - b) \\ &= 16S^2 , \end{aligned}$$

where S is the area of triangle ABC.

Now consider the numerator of h^2. Its value when $w = a^2 + b^2 - c^2$ is set equal to 0 is $4a^2b^2c^2 - a^2u^2 - b^2v^2 = 4a^2b^2c^2 - a^2(2b^2) - b^2(2a^2) = 0$, so that w divides the numerator. So also do u and v. Hence the numerator of degree 6 in a, b, c must be a multiple of uvw, also of degree 6 in a, b, c. Hence the numerator is a multiple of uvw. Comparing the coefficients of a^6 (say) gives that the numerator must be $2uvw$. Hence

$$h^2 = \frac{2uvw}{16S^2} = \frac{uvw}{8S^2} .$$

The volume V of the tetrahedron satisfies

$$V^2 = \left(\frac{Sh}{3}\right)^2 = \frac{S^2h^2}{9} = \frac{uvw}{72} ,$$

whence

$$V = \frac{\sqrt{uvw}}{6\sqrt{2}} = \frac{2uvw}{12} .$$

The checking for the tetrahedron proceeds as before.

194. Let ABC be a triangle with incentre I. Let M be the midpoint of BC, U be the intersection of AI produced with BC, D be the foot of the perpendicular from I to BC and P be the foot of the perpendicular from A to BC. Prove that

$$|PD||DM| = |DU||PM| \ .$$

Solution 1. Suppose that the lengths of the sides of the triangle are a, b and c, using the conventional notation. Then the distance from B of the following points on the side BC are given by $(B,0)$, (C,a), $(M,a/2)$, $(U,ca/(b+c))$, $(D,(a+c-b)/2)$ and $(P,c\cos B) = (P,(a^2+c^2-b^2)/(2a))$. One can then verify the desired relation by calculation.

Solution 2. [L. Chen] Let the side lengths of the triangle be a, b, c, as conventional, and, wolog, suppose that $c < b$. Let $u = |BP|$ and $v = |PC|$. Then, equating two expressions for the area of the triangle, with $r = |ID|$ as the inradius, we find that $|AP| = 2rs/a$. From similar triangle, we have that

$$\frac{|PU|}{|DU|} = \frac{|AP|}{|ID|} = \frac{2s}{a} = 1 + \frac{b+c}{a} \ ,$$

whence

$$\frac{|PD|}{|DU|} = \frac{b+c}{a} \ .$$

Now $|PM| = (a/2) - u = (v-u)/2$ and $|DM| = (b-c)/2$. Hence

$$\frac{|PM|}{|DM|} = \frac{v-u}{b-c} \ .$$

By Pythagoras' Theorem, $c^2 - u^2 = b^2 - v^2$, whence

$$\frac{v-u}{b-c} = \frac{b+c}{v+u} = \frac{b+c}{a}$$

and the result follows.

195. Let $ABCD$ be a convex quadrilateral and let the midpoints of AC and BD be P and Q respectively, Prove that

$$|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 = |AC|^2 + |BD|^2 + 4|PQ|^2 \ .$$

Solution 1. Let X denote the vector from an origin to a point X. Then, vectorially, it can be verified that

$$\begin{aligned}(A-B)\cdot(A-B) + (B-C)\cdot(B-C) + (C-D)\cdot(C-D) + (D-A)\cdot(D-A)\\ - (A-C)\cdot(A-C) - (B-D)\cdot(B-D)\\ = -2A\cdot B - 2B\cdot C - 2C\cdot D - 2D\cdot A + 2A\cdot C + 2B\cdot D + A^2 + B^2 + C^2 + D^2\\ = 4\left(\frac{A+C}{2} - \frac{B+D}{2}\right)\cdot\left(\frac{A+C}{2} - \frac{B+D}{2}\right) \ ,\end{aligned}$$

which yields the desired result.

Solution 2. [T. Yin] We use the result that for any parallelogram $KLMN$, $2|KL|^2 + 2|LM|^2 = |KM|^2 + |LN|^2$. This is straightforward to verify using the Law of Cosines, for example. Let W, X, Y, Z be the respective midpoints of the sides AB, BC, CD, DA. Using the fact that all of $WXYZ$, $PXQZ$ and $PWQY$ are parallelograms, we have that

$$\begin{aligned}|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 &= 4[|PX|^2 + |PW|^2 + |PZ|^2 + |PY|^2] \\ &= 2[|PQ|^2 + |XZ|^2 + |PQ|^2 + |WY|^2] \\ &= 4|PQ|^2 + 2[|XZ|^2 + |WY|^2] \\ &= 4|PQ|^2 + 4[|WZ|^2 + |WX|^2] \\ &= 4|PQ|^2 + |BD|^2 + |AC|^2 \ .\end{aligned}$$

196. Determine five values of p for which the polynomial $x^2 + 2002x - 1002p$ has integer roots.

Answer. Here are some values of $(p; u, v)$ with u and v the corresponding roots: $(0; 0, -2002)$, $(4; 2, -2004)$, $(784; 336, -2338)$, $(1780; 668, -2670)$, $(3004; 2002, -3004)$, $(3012; 1004, -3006)$, $(4460; 1338, -3340)$, $(8012; 2004, -4006)$, $(8024; 2006, -4008)$, $(-556; -334, -1668)$, $(-1000; -1000, -1002)$.

Solution 1. If x satisfies the equation $x^2 + 2002x - 1002p = 0$, then we must have $p = x(x+2002)/(1002)$. If we choose integers x for which $x(x + 2002)$ is a multiple of 1002, then this value of p will be an integer that yields a quadratic with two integer roots, namely x and $-2002 - x$. One way to do this is to select either $x \equiv 0$ or $x \equiv 2 \pmod{1002}$. Observing that $1002 = 2 \times 3 \times 167$, we can also try to make $x \equiv 0 \pmod{167}$ and $x \equiv 2 \pmod 6$. For example, $x = 668$ works. We can also try $x \equiv 2 \pmod{167}$ and $x \equiv 0 \pmod 6$; in this case, $x = 336$ works.

Solution 2. The discriminant of the quadratic is 4 times $1001^2 + 1002p$. Suppose that p is selected to make this equal to a square q^2. Then we have that

$$1002p = q^2 - 1001^2 = (q - 1001)(q + 1001) \ .$$

We select q so that either $q - 1001$ or $q + 1001$ is divisible by 1002. For example $q = 2003, 1, 3005, 4007$ all work. We can also make one factor divisible by 667 and the other by 6.

197. Determine all integers x and y that satisfy the equation $x^3 + 9xy + 127 = y^3$.

Solution 1. Let $x = y + z$. Then the equation becomes $(3z + 9)y^2 + (3z^2 + 9z)y + (z^3 + 127) = 0$, a quadratic in y whose discriminant is equal to

$$\begin{aligned}&(3z + 9)^2z^2 - 4(3z + 9)(z^3 + 127) \\ &\qquad = (3z + 9)[z^2(3z + 9) - 4(z^3 + 127)] \\ &\qquad = -(3z + 9)(z^3 - 9z^2 + 508) \ .\end{aligned}$$

Note that $z^3 - 9z^2 + 508 = z^2(z - 9) + 508$ is nonnegative if and only if $z \geq -5$ (z being an integer) and that $3z + 9$ is nonnegative if and only if $z \geq -3$. Hence the discriminant is nonnegative if and only if $z = -3, -4, -5$. ¿From the quadratic equation, we have that $z^3 + 127 \equiv 0 \pmod 3$. The only possibility is $z = -4$ and this leads to the equation $0 = -3y^2 + 12y + 63 = -3(y - 7)(y + 3)$ and the solutions $(x, y) = (3, 7), (-7, -3)$.

Solution 2. The equation can be rewritten

$$(x - y)[(x - y)^2 + 3xy] + 9xy = -127$$

or

$$u^3 + 3v(u + 3) = -127$$

where $u = x - y$ and $v = xy$. Hence

$$3v = -\frac{u^3 + 127}{u + 3} = -\left[(u^2 - 3u + 9) + \frac{100}{u+3}\right] .$$

Therefore, $u^3 + 127 \equiv 0 \pmod 3$, so that $u \equiv 2 \pmod 3$, and $u + 3$ divides 100. The candidates are

$$u = -103, -28, -13, -7, -4, -1, 2, 17, 47 .$$

Checking these out leads to the posible solutions.

198. Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that $f(0) = 0$ and $f(1) = 1$ and that, for every positive integer n, $f(n) \equiv 0$ or $f(n) \equiv 1$, modulo p. Prove that $d \geq p - 1$. Give an example of such a polynomial.

Solution. Since the polynomial is nonconstant, $d \geq 1$, so that the result holds for $p = 2$. Henceforth, assume that p is an odd prime. Let $0 \leq k \leq p - 2$. Consider the polynomial

$$p_k(x) = \frac{x(x-1)(x-2)\cdots(x-k+1)(x-k-1)\cdots(x-p+2)}{k!(p-k-2)!(-1)^{p-k}} .$$

We have that $p_k(k) = 1$ and $p_k(x) = 0$ when $x = 0, 1, 2, \cdots, k-1, k+1, \cdots, p-2$. Let

$$g(x) = \sum_{k=0}^{p-2} f(k) p_k(x) .$$

Then the degree of $g(x)$ does not exceed $p - 2$ and $g(x) = f(x)$ for $x = 0, 1, 2, \cdots, p-2$; in fact, $g(x)$ is the unique polynomial of degree less than $p - 1$ that agrees with f at these $p - 1$ points (why?).

Now

$$g(p-1) = \sum_{k=0}^{p-2} (-1)^{p-k} \frac{(p-1)!}{k!(p-k-1)!} f(k) = \sum_{k=0}^{p-2} (-1)^{p-k} \binom{p-1}{k} f(k) .$$

Since $\binom{p-1}{k} = \binom{p}{k} - \binom{p-1}{k-1}$ and $\binom{p}{k} \equiv 0 \pmod p$ for $1 \leq k \leq p-1$, and induction argument yields that $\binom{p-1}{k} \equiv (-1)^k$ for $1 \leq k \leq p-1$, so that

$$g(p-1) \equiv (-1)^p \sum_{k=0}^{p-2} f(k)$$

$\pmod p$. Since $f(0) = 0$ and $f(1) = 1$, it follows that $\sum_{k=0}^{p-2} f(k)$ is congruent to some number between 1 and $p-2$ inclusive, so that $g(p-1) \not\equiv 0$ and $g(p-1) \not\equiv 1 \pmod p$. Hence $f(p-1) \neq g(p-1)$, so that f and g are distinct polynomials. Thus, the degree of g exceeds $p - 2$ as desired.

By Fermat's Little Theorem, the polynomial x^{p-1} satisfies the condition.

Solution 2. [M. Guay-Paquet] Let

$$h(x) = f(x) + f(2x) + \cdots + f((p-1)x) .$$

Then $h(1) \not\equiv 0 \pmod p$ and $h(0) = 0$. The degree of h is equal to d, the degree of f.

Let $x \not\equiv 0 \pmod p$. Then $(x, 2x, 3x, \cdots, (p-1)x)$ is a permutation of $(1, 2, 3, \cdots, p-1)$, so that $h(x) \equiv h(1) \pmod p$.

Suppose that $g(x) = h(x) - h(1)$. The degree of g is equal to d, $g(0) \equiv -h(1) \not\equiv 0 \pmod p$ and $g(x) \equiv 0$ whenever $x \not\equiv 0 \pmod p$. Therefore, $g(x)$ differs from a polynomial of the form $k(x-1)(x-2)\cdots(x-\overline{p-1})$ by a polynomial whose coefficients are multiples of p. Since $k \not\equiv 0 \pmod p$ (check out the value at 0), the coefficient of x^{k-1} must be nonzero, and so $d \geq p - 1$, as desired.