Solutions to problems

103. Determine a value of the parameter θ so that

$$f(x) \equiv \cos^2 x + \cos^2(x+\theta) - \cos x \cos(x+\theta)$$

is a constant function of x.

Solution 1.

$$\begin{aligned} f(x) &= \cos^2 x + (\cos x \cos\theta - \sin x \sin\theta)^2 - \cos x(\cos x \cos\theta - \sin x \sin\theta) \\ &= \cos^2 x(1 + \cos^2\theta - \cos\theta) + (1 - \cos^2 x)(\sin^2\theta) - \sin x \cos x \sin\theta(2\cos\theta - 1) \\ &= \sin^2\theta + \cos^2 x(1 + \cos^2\theta - \cos\theta - 1 + \cos^2\theta) - \frac{1}{2}\sin 2x \sin\theta(2\cos\theta - 1) \\ &= \sin^2\theta + (2\cos\theta - 1)(\cos^2 x \cos\theta - \sin 2x \sin\theta) \ . \end{aligned}$$

The function $f(x)$ is constant when $2\cos\theta - 1 = 0$, or when $\theta = \pi/3$, and its constant value in this case is $3/4$.

Solution 2.

$$\begin{aligned} f(x) &= \frac{1 + \cos 2x}{2} + \frac{1 + \cos 2(x+\theta)}{2} - \frac{1}{2}(\cos(2x+\theta) - \cos\theta) \\ &= \frac{1}{2}[2 - \cos\theta + \cos 2x(1 + \cos 2\theta - \cos\theta) + \sin 2x(\sin\theta - \sin 2\theta)] \\ &= \frac{1}{2}[2 - \cos\theta + (2\cos\theta - 1)(\cos 2x \cos\theta + \sin 2x \sin\theta] \ . \end{aligned}$$

When $\theta = \pi/3$, $\cos\theta = 1/2$ and the function is the constant $3/4$.

Solution 3. First, note the identity

$$\cos^2 A + \cos^2 B = 1 + \frac{1}{2}(\cos 2A + \cos 2B) = 1 + \cos(A+B)\cos(A-B) \ .$$

Applying this yields

$$\begin{aligned} f(x) &= 1 + \cos(2x+\theta)\cos\theta - \frac{1}{2}(\cos(2x+\theta) + \cos\theta) \\ &= \left(1 - \frac{1}{2}\cos\theta\right) + \cos(2x+\theta)\left[\cos\theta - \frac{1}{2}\right] \ . \end{aligned}$$

Hence, $f(x)$ is the constant $3/4$ when $\theta = \pi/3$.

Solution 4. We have that

$$f(x) = \frac{3}{4}\cos^2 x + [\cos(x+\theta) - \frac{1}{2}\cos x]^2 \ .$$

The function $f(x)$ can be made to take the constant value $3/4$ if we can find a parameter θ for which

$$[\cos(x+\theta) - \frac{1}{2}\cos x]^2 = \frac{3}{4}\sin^2 x$$

for all x. This is equivalent to

$$\cos(x+\theta) = \frac{1}{2}\cos x \pm \frac{\sqrt{3}}{2}\sin x = \cos(x \pm \frac{\pi}{3})$$

for all x. Thus, if $\theta = \pm\pi/3$, then $f(x)$ is constantly equal to $\frac{3}{4}$.

Comment. Some students started by showing that $f(0) = f(\frac{\pi}{2})$ implies that $1 + \cos^2\theta - \cos\theta = \cos^2(\frac{\pi}{2} + \theta) = 1 - \cos^2\theta$, or $0 = 2\cos^2\theta - \cos\theta = \cos\theta(2\cos\theta - 1)$. This tells us that $\theta \equiv \frac{\pi}{2} \pmod{\pi}$ and $\theta \equiv \pm\frac{\pi}{3} \pmod{2\pi}$ are the only possibilities. However, the first of these turns out to be extraneous. It yields $f(x) = 1 \pm \frac{1}{2}\sin 2x$, which is not constant.

104. Prove that there exists exactly one sequence $\{x_n\}$ of positive integers for which

$$x_1 = 1 , \quad x_2 > 1 , \qquad x_{n+1}^3 + 1 = x_n x_{n+2}$$

for $n \geq 1$.

Solution. Let $x_2 = u$. Then the first four terms of the sequence are

$$1, \quad u, \quad u^3 + 1, \qquad u^8 + 3u^5 + 3u^2 + (2/u), \cdots$$

so for the whole sequence to consist of positive integers, we must have that $u = 2$. Now for any $n \geq 3$,

$$x_n = \frac{x_{n-1}^3 + 1}{x_{n-2}} = \frac{x_{n-2}^9 + 3x_{n-2}^6 + 3x_{n-2}^3 + 1 + x_{n-3}^3}{x_{n-2}x_{n-3}^3} \qquad (1).$$

¿From the given condition, it can be seen that any consecutive pairs of terms in the sequence, if integers, are coprime. We know that x_1, x_2, x_3 are integers. Let $n \geq 4$. Suppose that it has been shown that x_m is an integer for $1 \leq m \leq n-1$. Then $(x_{n-3}^3 + 1)/x_{n-2} = x_{n-4}$ is an integer, as is $(x_{n-2}^3 + 1)/x_{n-3} = x_{n-1}$ and its cube. Since the numerator of (1) is a multiple of each of x_{n-2} and x_{n-3}^3 separately, and since these two divisors are coprime, x_n must be an integer. The result follows by induction.

Solution 2. [M. Mika] As before, we see that x_2 must be 2. It can be checked that x_3 and x_4 are integers. For any integer $n \geq 4$, we have that

$$\begin{aligned} x_n^3 + 1 &= \frac{(x_{n-1}^3 + 1)^3}{x_{n-2}^3} + 1 \\ &= \frac{(x_{n-1}^3 + 1)^3}{x_{n-1}x_{n-3} - 1} + 1 \\ &= \frac{x_{n-1}(x_{n-1}^8 + 3x_{n-1}^5 + 3x_{n-1}^2 + x_{n-3})}{x_{n-1}x_{n-3} - 1} \ . \end{aligned}$$

Therefore

$$(x_n^3 + 1)(x_{n-1}x_{n-3} - 1) = x_{n-1}(x_{n-1}^8 + 3x_{n-1}^5 + 3x_{n-1}^2 + x_{n-3}) \ .$$

Supposing, as an induction hypothesis,that $x_1, \cdots, x_n$ are integers, we see that x_{n-1} and $x_{n-1}x_{n-3} - 1$ are coprime, so we must have that x_{n-1} divides $x_n^3 + 1$. Thus, x_{n+1} is an integer.

105. Prove that within a unit cube, one can place two regular unit tetrahedra that have no common point.

Solution 1. Let $ABCDEFGH$ be the cube, with $ABCD$ the top face, $EFGH$ the lower face and AE, BF, CG, DH edges. Let O be the centre of the cube, and let P, Q, R be the midpoints of AB, DH, FG respectively.

The centre O lies on the diagonal CE, which is the axis of a rotation that takes $B \rightarrow D \rightarrow G$, $A \rightarrow H \rightarrow F$, $P \rightarrow Q \rightarrow R$, so ΔPQR is equilateral with centre O and $CE \perp PQR$.

Using Pythagoras' Theorem, we calculate some lengths:

$$|CP| = |CQ| = |CR| = |RB| = |RH| = \sqrt{1 + (1/4)} = (\sqrt{5})/2 \ ,$$

$$|PQ| = |PR| = |QR| = \sqrt{(5/4) + (1/4)} = \sqrt{3/2}\ ,$$
$$|CO| = (\sqrt{3})/2\ .$$
[As a check that O is the centre of ΔPQR, we can compute $|PO| = |QO| = |RO| = 1/\sqrt{2} = [1/\sqrt{3}]|PQ|$.]

Since the height of a regular tetrahedron with side s is $s\sqrt{2/3}$, we can construct a regular tetrahedron $CUVW$ with apex C, base UVW homothetic to PQR with centre O, height $(\sqrt{3})/2$, and side length $[(\sqrt{3})/2][\sqrt{3/2}] = 3/(2\sqrt{2}) > 1$. Since $3/(2\sqrt{2}) = \sqrt{9/8} < \sqrt{3/2}$, triangle UVW lies within triangle PQR, and so the tetrahedron lies within the cube. Shrink the tetrahedron by a homothety with factor $\sqrt{8/9}$ about its centre to get one of the desired tetrahedra.

The second tetrahedra can be found in a similar way from $EUVW$ (which is congruent to $CUVW$). The two tetrahedra are strictly separated by the plane of PQR.

Solution 2. Let the cube have vertices at the eight points (ϵ, η, ζ) where $\epsilon, \eta, \zeta = 0, 1$. The plane of equation $x + y + z = 3/2$ passes through $(0, 1, \frac{1}{2})$, $(\frac{1}{2}, 1, 0)$, $(1, \frac{1}{2}, 0)$, $(1, 0, \frac{1}{2})$, $(\frac{1}{2}, 0, 1)$ and $(0, \frac{1}{2}, 1)$ at the middle of various edges of the cube, and bisects the cube into two congruent halves. Consider the cube reduced by a homothety of factor $1/\sqrt{2}$ about the origin. Four of its vertices, $(0, 0, 0)$, $(0, 1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, 0, 1/\sqrt{2})$, constitute the four vertices of a regular unit tetrahedron contained in the original cube. Since the sum of the coordinates of all of these points is less than $3/2$, they all lie on the same side of the plane bisecting the cube, as does the whole tetrahedron. Its image reflected in the centre of the cube is a second tetrahedron contained in the upper portion of the cube.

Solution 3. [D. Tseng] Consider unit tetrahedra $CPQR$ and $EUVW$, each sharing a vertex and a face with the cube and directed inwards with the face diagonal AC intersecting PQ in its midpoint S and face diagonal EG intersecting UV in its midpoint X. (Each tetrahedron is carried into the other by a reflection in the centre of the cube.) Planes PQR and UVW are parallel. These tetrahedra intersect the internal plane $ACGE$ in two triangles EXW and CSR. Let SR produced meet EG in M, and let T and N be be points on AC for which $RT \perp AC$ and $MN \perp AC$. Observe that $|RS| = |CS| = \sqrt{3}/2$, $|ST| = 1/(2\sqrt{3})$, $|TC| = 1/\sqrt{3}$ and $|RT| = \sqrt{(2/3)}$. Since $ST : SN = RT : MN$, $|SN| = 1/(2\sqrt{2})$ and $|MG| = |NC| = ((\sqrt{3})/2) - (1/(2\sqrt{2}))$. Hence $|EX| + |MG| = \sqrt{3} - 1/(2\sqrt{2}) < \sqrt{2} = |EG|$. This means that the parallel lines WX and RS have a region of $ACGE$ between them that do not intersect triangles EXW and CSR, and so the two tetrahedra are separated by the slab between the parallel planes PQR and UVW that passes through the centre of the cube.

106. Find all pairs (x, y) of positive real numbers for which the least value of the function

$$f(x, y) = \frac{x^4}{y^4} + \frac{y^4}{x^4} - \frac{x^2}{y^2} - \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x}$$

is attained. Determine that minimum value.

Solution 1. Observe that

$$f(x, y) - 2 = \left(\frac{x^2}{y^2} - 1\right)^2 + \left(\frac{y^2}{x^2} - 1\right)^2 + \left(\frac{x}{y} - \frac{y}{x}\right)^2 + \left(\frac{x}{y} - 2 + \frac{y}{x}\right) \geq \frac{(x-y)^2}{xy} \geq 0$$

with equality if and only if $x = y$. The required minimum is 2.

Solution 2. Let $u = (x^2/y^2) + (y^2/x^2)$ and $v = (x/y) + (y/x)$ Note that $u, v \geq 2$ with equality if and only if $x = y$. Then $f(x, y) = u^2 - u - 2 + v = (u - 2)(u + 1) + v \geq 2$ with equality if and only if $x = y$. The desired minimum is 2.

Solution 3. Let $v = (x/y) + (y/x)$. Then

$$f(x, y) = [v^2 - 2]^2 - 2 - [v^2 - 2] + v = v^4 - 5v^2 + v + 4 = (v - 2)(v^3 + 2v^2 - v - 1) + 2\ .$$

Note that $v \geq 2$ and so $v^3 + 2v^2 - v - 1 = (v^3 - v) + (2v^2 - 1) > 0$. The desired result now follows.

Comment. Several solvers did this problem by calculus. The most important thing you need to know about calculus is when not to use it. Calculus provides very general algorithms for doing optimization problems, and such algorithms often have two undesirable characteristics: (1) they may not provide the quickest and most convenient approach in particular cases; (2) they tend to operate as "black boxes", preventing the solver from appreciating the essence of the problem or the significance of the answer. When you have a problem of this type, you should check to see whether it can be handled without calculus, and use calculus only as a last resort, or at least when it is clear that every other approach is messier.

107. Given positive numbers a_i with $a_1 < a_2 < \cdots < a_n$, for which permutation $(b_1, b_2, \cdots, b_n)$ of these numbers is the product

$$\prod_{i=1}^{n} \left(a_i + \frac{1}{b_i} \right)$$

maximized?

Solution 1. By the arithmetic-geometric means inequality, we have that, for each i, $2a_ib_i \leq a_i^2 + b_i^2$, so that

$$(a_ib_i + 1)^2 = a_i^2b_i^2 + 2a_ib_i + 1 \leq a_i^2b_i^2 + a_i^2 + b_i^2 + 1 = (a_i^2 + 1)(b_i^2 + 1) \ .$$

Hence

$$\prod_{i=1}^{n}(a_ib_i + 1) \leq \sqrt{\prod_{i=1}^{n}(a_i^2 + 1)\prod_{i=1}^{n}(b_i^2 + 1)} = \prod_{i=1}^{n}(a_i^2 + 1) \ .$$

Equality occurs if and only if $b_i = a_i$ for each i.

Now

$$\prod_{i=1}^{n} \left(a_i + \frac{1}{b_i} \right) = \frac{\prod_{i=1}^{n}(a_ib_i + 1)}{\prod_{i=1}^{n} b_i} = \frac{\prod_{i=1}^{n}(a_ib_i + 1)}{\prod_{i=1}^{n} a_i} \ .$$

Thus, the given expression is maximized $\Leftrightarrow \prod_{i=1}^{n}(a_ib_i+1)$ is maximized $\Leftrightarrow a_i = b_i$ for each $i \Leftrightarrow (b_1, b_2, \cdots, b_n)$ is obtained from $(a_1, a_2, \cdots, a_n)$ by the identity permutation.

Solution 2. There are finitely many permutations of the numbers, so that there must be a permutation which maximizes the value of the given expression. We show that it is the identity permutation, by showing that, for any other permutation, we can find a permutation that yields a larger value.

Suppose that $(b_1, b_2, \cdots, b_n)$ is a permutation for which there is a pair i, j of indices for which $a_i < a_j$ while $b_i > b_j$. Then

$$\left(a_i + \frac{1}{b_j} \right)\left(a_j + \frac{1}{b_i} \right) - \left(a_i + \frac{1}{b_i} \right)\left(a_j + \frac{1}{b_j} \right) = (a_j - a_i)\left(\frac{1}{b_j} - \frac{1}{b_i} \right) > 0 \ .$$

with the result that the product can be made larger by interchanging the positions of b_i and b_j. The result follows.

108. Determine all real-valued functions $f(x)$ of a real variable x for which

$$f(xy) = \frac{f(x) + f(y)}{x + y}$$

for all real x and y for which $x + y \neq 0$.

Solution 1. Setting $y = 1$ yields that $(x+1)f(x) = f(x) + f(1)$ so that $xf(x) = f(1)$ for $x \neq -1$. Set $x = 0$ to obtain $f(1) = 0$, so that, for $x \neq 0$, $(x+1)f(x) = f(x)$. ¿From this, we deduce that, as long as $x \neq 0, -1$, we have that $f(x) = 0$.

For each nonzero value of x, $xf(0) = f(x) + f(0)$, so that $(x-1)f(0) = f(x)$. Taking $x = 2$ gives $f(0) = f(2) = 0$. Finally, $2f(-1) = -2f(1) = 0$, so $f(-1) = 0$. Hence, $f(x)$ must be indentically equal to 0.

Solution 2. [S.E. Lu] For all nonzero x, we have that $f(x) = (x-1)f(0)$. The equality $f(x) + f(y) = (x+y)f(xy)$ leads to either $f(0) = 0$ or $x+y-2 = (xy-1)(x+y)$. The latter simplifies to $(x+y)(2-xy) = 2$ for all nonzero x, y, which is patently false. Hence $f(0) = 0$, so $f(x) \equiv 0$.

Solution 3. Taking $y = 0$ leads to $(x-1)f(0) = f(x)$ for all $x \neq 0$. Taking $y = 1$ leads to $xf(x) = f(1)$ for all $x \neq -1$. Hence, for $x \neq 0, 1$, we have that $x(x-1)f(0) = f(1)$. This holds for infinitely many x if and only if $f(0) = f(1) = 0$. It follows that $f(x) = 0$ for all real x.

Comment. Suppose that the given condition is weakened to hold only when *both* x and y are nonzero. Then we get $xf(x) = f(1)$ so that $f(x) = f(1)/x$ for all nonzero x. it can be checked that, for any constant c, $f(x) = c/x$ for is a solution for $x \neq 0$.