Solutions

109. Suppose that

$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}=k\ .$$

Find, in terms of k, the value of the expression

$$\frac{x^8+y^8}{x^8-y^8}+\frac{x^8-y^8}{x^8+y^8}\ .$$

Solution 1. Simplifying, we obtain that

$$k=\frac{2(x^4+y^4)}{x^4-y^4}\ ,$$

and, by extension, that

$$\frac{k^2+4}{2k}=\frac{k}{2}+\frac{2}{k}=\frac{x^4+y^4}{x^4-y^4}+\frac{x^4-y^4}{x^4+y^4}=\frac{2(x^8+y^8)}{x^8-y^8}\ .$$

Continuing on, we find that

$$\frac{x^8+y^8}{x^8-y^8}+\frac{x^8-y^8}{x^8+y^8}=\frac{2(x^{16}+y^{16})}{x^{16}-y^{16}}=\frac{k^2+4}{4k}+\frac{4k}{k^2+4}=\frac{k^4+24k^2+16}{4k(k^2+4)}\ .$$

Comment. R. Barrington Leigh defined a formula

$$R=\frac{a+b}{a-b}+\frac{a-b}{a+b}=2\cdot\frac{a^2+b^2}{a^2-b^2}$$

for $a\neq\pm b$ which he then applied to $(a,b)=(x^2,y^2),(x^4,y^4)$.

110. Given a triangle ABC with an area of 1. Let $n>1$ be a natural number. Suppose that M is a point on the side AB with $AB=nAM$, N is a point on the side BC with $BC=nBN$, and Q is a point on the side CA with $CA=nCQ$. Suppose also that $\{T\}=AN\cap CM$, $\{R\}=BQ\cap AN$ and $\{S\}=CM\cap BQ$, where $\cap$ signifies that the singleton is the intersection of the indicated segments. Find the area of the triangle TRS in terms of n.

Solution 1. [R. Furmaniak, Y. Ren] The area of a triangle XYZ will be denoted by $[XYZ]$. Consider the triangle ABQ and the line MC that intersects AB at M, BQ at S and AQ at the external point C. By Menelaus' Theorem for the triangle ABQ and transversal MC,

$$1=\frac{BM}{MA}\cdot\frac{AC}{QC}\cdot\frac{QS}{SB}=(n-1)n\frac{QS}{SB}\Longrightarrow\frac{SB}{QS}=(n-1)n\ .$$

Observe the triangles BSC and QSC. Since the heights from C to the opposite sides SB and QS coincide, then

$$\frac{[BSC]}{[QSC]}=\frac{SB}{QS}=(n-1)n\Longrightarrow[BSC]=(n-1)n[QSC]\ .$$

Examining triangles QSC and QBC, we similarly find that

$$\frac{[QSC]}{[QBC]}=\frac{QS}{QB}=\frac{QS}{QS+SB}=\frac{1}{(n-1)n+1}=\frac{1}{n^2-n+1}$$

$$\Longrightarrow [QSC] = \frac{1}{n^2-n+1} \cdot [QBC] \; .$$

Since the heights of triangles QBC and ABC from B to QC and AC coincide, it follows that

$$\frac{[QBC]}{[ABC]} = \frac{QC}{AB} = \frac{1}{n} \Longrightarrow [QBC] = \frac{1}{n} \cdot [ABC] = \frac{1}{n}$$

$$\Longrightarrow [QSC] = \frac{1}{n^2-n+1} \cdot \frac{1}{n} = \frac{1}{n(n^2-n+1)}$$

$$\Longrightarrow [BSC] = \frac{(n-1)n}{n(n^2-n+1)} = \frac{n-1}{n^2-n+1} \; .$$

Similarly,

$$[TAC] = [RBA] = \frac{n-1}{n^2-n+1} \; .$$

Now

$$[TSR] = [ABC] - [BSC] - [TAC] - [RBA] = 1 - \frac{3(n-1)}{n^2-n+1} = \frac{(n-2)^2}{n^2-n+1} \; .$$

Solution 2. [M. Butler] Using Menelaus' Theorem with triangle BQC and transversal ARN, we find that

$$\frac{BR}{RQ} \cdot \frac{QA}{AC} \cdot \frac{CN}{NB} = -1$$

so that

$$\frac{BR}{RQ} \cdot \frac{1-n}{n} \cdot \frac{n-1}{1} = -1$$

$$\Longrightarrow BR = \frac{n}{(n-1)^2} \cdot RQ \; .$$

Thus,

$$\begin{aligned} \frac{n-1}{n} &= [ABQ] = [ARB] + [ARQ] \\ &= \left[1 + \frac{(n-1)^2}{n}\right][ARB] \\ &= \left[\frac{n^2-n+1}{n}\right][ARB] \end{aligned}$$

whence

$$[ARB] = \frac{n-1}{n^2-n+1} \; .$$

Therefore

$$[RST] = 1 - \frac{3(n-1)}{n^2-n+1} = \frac{(n-2)^2}{n^2-n+1} \; .$$

Solution 3. Let $a = [AMT]$, $b = [BNR]$, $c = [CQS]$, $x = [BRT]$, $y = [CSR]$, $z = [ATS]$ and $d = [RST]$. Then using $BM = (n-1)AM$, $[BRM] = (n-1)a$,

$$x + d = [BTS] = (n-1)[ATS] = (n-1)z$$

and

$$nb + y = [BRC] + [CSR] = [BSC] = (n-1)[ASC] = (n-1)nc \; .$$

Analogously, from $CN = (n-1)BN$ and $AQ = (n-1)CQ$, we get

$$x + d = (n-1)z, \quad y + d = (n-1)x, \quad z + d = (n-1)y$$

and

$$nb + y = (n^2 - n)c, \quad nc + z = (n^2 - n)a, \quad na + x = (n^2 - n)b \ ,$$

whence

$$x + y + x + 3d = (n-1)(x + y + z) \quad \text{or} \quad 3d = (n-2)(x+y+z)$$

and

$$n(a+b+c) + (x+y+z) = (n^2 - n)(a+b+c) \quad \text{or} \quad x+y+z = (n^2-2n)(a+b+c) \ .$$

¿From $1 = n[na + x + b] = n[nb + y + c] = n[nc + z + a]$, we find that

$$\begin{aligned} 3 &= n^2(a+b+c) + n(x+y+z) + n(a+b+c) \\ &= n(n+1)(a+b+c) + n(x+y+z) \\ &= \left[\frac{n+1}{n-2} + n\right](x+y+z) \\ &= \left[\frac{n^2-n+1}{n-2}\right](x+y+z) \ , \end{aligned}$$

whence

$$d = \frac{(n-2)^2}{n^2 - n + 1} \ .$$

Solution 4. Since the ratio of areas remains invariant under shear transformations and dilations, we may assume that the triangle is right isosceles. Assign coordinates: $A \sim (0,0)$, $B \sim (0,1)$, $C = (1,0)$, $M \sim (0, 1/n)$,

$$N \sim \left(\frac{1}{n}, \frac{n-1}{n}\right)$$

$$Q \sim \left(\frac{n-1}{n}, 0\right) \ .$$

Then

$$R \sim \left(\frac{n-1}{n^2-n+1}, \frac{(n-1)^2}{n^2-n+1}\right)$$

$$S \sim \left(\frac{(n-1)^2}{n^2-n+1}, \frac{1}{n^2-n+1}\right)$$

and

$$T \sim \left(\frac{1}{n^2-n+1}, \frac{n-1}{n^2-n+1}\right) \ .$$

A computation of the area of ΔRST now yields the result.

Comment. Considering how reasonable the result it, H. Lee noted that when $n > 1$, then $(n-2)^2 < n^2 - n + 1$, so that $[TSR] < 1$ as expected, and also noted that when $n = 2$, we get the special case of the medians that intersect in a common point and yield $[TSR] = 0$.

111. (a) Are there four different numbers, not exceeding 10, for which the sum of any three is a prime number?

(b) Are there five different natural numbers such that the sum of every three of them is a prime number?

Solution. [R. Barrington Leigh] (a) Yes, there are four such numbers. The three-member sums of the set $\{1, 3, 7, 9\}$ are the primes 11, 13, 17, 19.

(b) No. We prove the statement by contradiction. Suppose that there are five different natural numbers for which every sum of three is prime. As the numbers are distinct and positive, each such sum must be at least $1 + 2 + 3 = 6$, and so cannot be a multiple of 3. Consider the five numbers, modulo 3. If there are three in the same congruent class, their sum is a multiple of 3. If there is one each congruent to 0, 1, 2 modulo 3, then the sum of these three is a multiple of 3. Otherwise, there are only two congruence classes represented with at most two numbers in each, an impossibility. Hence in all cases, there must be three who sum to a multiple of 3.

Comment. Part (b) need not be framed as a contradiction. One could formulate it as follows: Let five positive integers be given. Argue that either each congruent class modulo 3 is represented or that some class is represented by at least three of the numbers. Then note that therefore some three must sum to a multiple of 3. Observe that 3 itself is not a possible sum. Hence, among every five positive integers, there are three who add to a nonprime multiple of 3, and simply say that the answer to the question is "no".

112. Suppose that the measure of angle BAC in the triangle ABC is equal to α. A line passing through the vertex A is perpendicular to the angle bisector of $\angle BAC$ and intersects the line BC at the point M. Find the other two angles of the triangle ABC in terms of α, if it is known that $BM = BA + AC$.

Solution. Let q be the line through A perpendicular to the bisector of angle BAC; this line bisects the external angle at A. The possibility that q is parallel to BC is precluded by the condition that it intersects BC at M. Let $\angle ABC = \beta$ and $\angle ACB = \gamma$, Since p is not parallel to BC, β is not equal to γ.

Case i. Suppose that $\beta > \gamma$. Then M intersects BC so that B lies between M and C. Let BA be produced to D so that $AC = AD$. Since $MB = BA + AC = BA + AD = BD$, $\angle DMB = \angle MDB$ and so $\angle MDB = \frac{1}{2}\angle DBC = \frac{1}{2}\beta$ (exterior angle). Since $AD = AC$, $\angle ADC = \angle ACD = \frac{1}{2}\angle BAC = \frac{1}{2}\alpha$. Since MA produced bisects $\angle DAC$, MA produced right bisects DC and so $MD = MC$. Therefore

$$\gamma + \frac{\alpha}{2} = \angle MCD = \angle MDC = \frac{\beta}{2} + \frac{\alpha}{2} ,$$

whence $\beta = 2\gamma$. Therefore, $180° = \alpha + \beta + \gamma = \alpha + 3\gamma$ and

$$\gamma = \frac{180° - \alpha}{3} \qquad \text{and} \qquad \beta = \frac{360° - 2\alpha}{3} .$$

Case ii. Supppose that $\beta < \gamma$. Then M intersects BC so that C lies between B and M. Let BA be produced to D so that $AD = AC$. Since AM bisects $\angle DAC$, it right bisects CD and so triangle MDC is isosceles. Then $\angle ADM = \angle ADC + \angle MDC = \angle ACD + \angle MCD = \angle ACM$. Since $BM = BA + AC = BD$, $\angle ACM = \angle ADM = \angle BDM = \angle BMD$. Since $\angle ACM = \alpha + \beta$ (exterior angle), $180° = \angle DBM + 2\angle BMD = \beta + 2(\alpha + \beta)$, so that $\beta = \frac{1}{3}(180° - 2\alpha)$ and $\gamma = 180° - \angle ACM = 180° - (\alpha + \beta) = 120° - \frac{1}{3}\alpha = (360° - \alpha)/3$.

Question. Why cannot you just say the second case can be handled as the first case, by symmetry?

113. Find a function that satisfies all of the following conditions:

(a) f is defined for every positive integer n;

(b) f takes only positive values;

(c) $f(4) = 4$;

(d)

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)} .$$

Solution. [R. Barrington Leigh] The function for which $f(n) = n$ for every positive integer n satisfies the condition. [*Exercise:* establish this by induction.] We now show that this is the only example. Substituting $n = 1$ into (d) and noting that $f(2) \neq 0$, we find that $f(1)^2 = 1$, whence $f(1) = 1$. Applying (d) to two consecutive values of the argument yields that

$$\frac{f(n)}{f(n+1)} - \frac{f(n-1)}{f(n)} = \frac{1}{f(n)f(n+1)} ,$$

whence

$$[f(n)]^2 - 1 = f(n-1)f(n+1) \ .$$

Substituting $n = 2$ and $n = 3$ into this and noting that $f(1) = 1$ and $f(4) = 4$, we find that

$$[f(2)]^2 - 1 = f(3)$$

and

$$[f(3)]^2 - 1 = 4f(2) \ ,$$

whence

$$0 = [f(2)]^4 - 2[f(2)]^2 - 4f(2) = f(2)[f(2) - 2]([f(2)]^2 + 2f(2) + 2) \ .$$

Since the first and third factors are positive for all postive possibilities for $f(2)$, we must have $f(2) = 2$. As

$$f(n+1) = \frac{[f(n)]^2 - 1}{f(n-1)} \ ,$$

we can prove by induction that $f(n) = n$ for all positive integers n.

114. A natural number is a multiple of 17. Its binary representation (*i.e.*, when written to base 2) contains exactly three digits equal to 1 and some zeros.

(a) Prove that there are at least six digits equal to 0 in its binary representation.

(b) Prove that, if there are exactly seven digits equal to 0 and three digits equal to 1, then the number must be even.

Solution 1. (a) If there are fewer than six digits equal to 0 in its binary representation, then the number must have at most eight digits and be of the form $2^a + 2^b + 2^c$ where $0 \le a < b < c \le 7$. The first eight powers of 2 with nonnegative exponent are congruent to $1, 2, 4, 8, -1, -2, -4, -8$ modulo 17, and the sum of any three of these cannot equal to zero and must lie between -14 and 14. Hence it is not possible for three powers of 2 among the first eight to sum to a multiple of 17. Hence, the number must have at least nine digits, including three zeros.

(b) Suppose that the number is equal to $2^a + 2^b + 2^c$ where $0 \le a < b < c \le 9$. If this number has exactly 10 digits and is odd, then $a = 0$ and $c = 9$, so that the number is equal to $1 + 2^b + 2^9 = 513 + 2^b \equiv 3 + 2^b \pmod{17}$. But there is no value of b that will make this vanish, modulo 17. Hence, a 10-digit number divisible by 17 must be even. An example is $2 \times 17^2 = 2 \times (1 + 2^5 + 2^8) = (1001000010)_2$.

Solution 2. [R. Furmaniak] (a) Since $17_{10} = 10001_2$, any binary number $abcd_2$ with four or fewer digits multiplied by 17 will yield $abcdabcd_2$. Since the first and last four digits are the same, there must be an even number of 1s. Thus, any multiple of 17 with exactly three binary digits must be a product of 17 and a number that has at least 5 binary digits. Every such product must have at least 9 digits, and so at least three digits equal to 0.

(b) As in Solution 1.