PROBLEMS FOR MAY

Please send your solutions to
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no later than **June 30, 2001**

Notes. A set in any space is *convex* if and only if, given any two points in the set, the line segment joining them is also contained in the set. A *closed* set is one that contains its boundary. A real sequence $\{x_n\}$ *converges* if and only if there is a number c, called its *limit*, such that, as n increases, the number x_n gets closer and closer to c. If the sequences is *increasing* (*i.e.*, $x_{n+1} \geq x_n$ for each index n) and *bounded above* (*i.e.*, there is a number M for which $x_n \leq M$ for each n, then it must converge. [Do you see why this is so?] Similarly, a decreasing sequence that is bounded below converges. [Supply the definitions and justify the statement.] An infinite *series* is an expression of the form $\sum_{k=a}^{\infty} x_k = x_a + x_{a+1} + x_{a+2} + \cdots + x_k + \cdots$, where a is an integer, usually 0 or 1. The nth *partial sum* of the series is $s_n \equiv \sum_{k=a}^{n} x_k$. The series has sum s if and only if its sequence $\{s_n\}$ of partial sums converges and has limit s; when this happens, the series *converges*. If the sequence of partial sums fails to converge, the series *diverges*. If every term in the series is nonnegative and the sequence of partial sums is bounded above, then the series converges. If a series of nonnegative terms converges, then it is possible to rearrange the order of the terms without changing the value of the sum.

79. Let x_0, x_1, x_2 be three positive real numbers. A sequence $\{x_n\}$ is defined, for $n \geq 0$ by

$$x_{n+3} = \frac{x_{n+2} + x_{n+1} + 1}{x_n} \ .$$

Determine all such sequences whose entries consist solely of positive integers.

80. Prove that, for each positive integer n, the series

$$\sum_{k=1}^{\infty} \frac{k^n}{2^k}$$

converges to twice an odd integer not less than $(n+1)!$.

81. Suppose that $x \geq 1$ and that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and the fractional part $\{x\}$ satisfies $0 \leq x < 1$. Define

$$f(x) = \frac{\sqrt{\lfloor x \rfloor} + \sqrt{\{x\}}}{\sqrt{x}} \ .$$

(a) Determine the small number z such that $f(x) \leq z$ for each $x \geq 1$.

(b) Let $x_0 \geq 1$ be given, and for $n \geq 1$, define $x_n = f(x_{n-1})$. Prove that $\lim_{n\to\infty} x_n$ exists.

82. (a) A regular pentagon has side length a and diagonal length b. Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3 \ .$$

(b) A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove that:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

83. Let C be a circle with centre O and radius 1, and let F be a closed convex region inside C. Suppose from each point C, we can draw two rays tangent to F meeting at an angle of $60°$. Describe F.

84. Let ABC be an acute-angled triangle, with a point H inside. Let U, V, W be respectively the reflected image of H with respect to axes BC, AC, AB. Prove that H is the orthocentre of ΔABC if and only if U, V, W lie on the circumcircle of ΔABC,