PROBLEMS FOR JUNE

Please send your solutions to
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no later than **July 31, 2001**.

85. Find all pairs (a, b) of positive integers with $a \neq b$ for which the system

$$\cos ax + \cos bx = 0$$

$$a \sin ax + b \sin bx = 0$$

has a solution. If so, determine its solutions.

86. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $CB = CD$. Prove that

(a) it is possible to inscribe a circle in it;

(b) it is possible to circumscribe a circle about it if and only if $AB \perp BC$;

(c) if $AB \perp AC$ and R and r are the respective radii of the circumscribed and inscribed circles, then the distance between the centres of the two circles is equal to the square root of $R^2 + r^2 - r\sqrt{r^2 + 4R^2}$.

87. Prove that, if the real numbers a, b, c, satisfy the equation

$$\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor$$

for each positive integer n, then at least one of a and b is an integer.

88. Let I be a real interval of length $1/n$. Prove that I contains no more than $\frac{1}{2}(n+1)$ irreducible fractions of the form p/q with p and q positive integers, $1 \leq q \leq n$ and the greatest common divisor of p and q equal to 1.

89. Prove that there is only one triple of positive integers, each exceeding 1, for which the product of any two of the numbers plus one is divisible by the third.

90. Let m be a positive integer, and let $f(m)$ be the smallest value of n for which the following statement is true:

given any set of n integers, it is always possible to find a subset of m integers whose sum is divisible by m

Determine $f(m)$.