International Mathematical Talent Search – Round 43

Problem 1/43. We will say that a rearrangement of the letters of a word *has no fixed letters* if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, the blocks of letters below shows that $E\,S\,A\,R\,E\,T$ is a rearrangement with no fixed letters of $T\,E\,R\,E\,S\,A$, but $R\,E\,A\,S\,T\,E$ is not.

$$\begin{array}{cccccc} T & E & R & E & S & A \\ E & S & A & R & E & T \end{array} \qquad\qquad \begin{array}{cccccc} T & E & R & E & S & A \\ R & E & A & S & T & E \end{array}$$

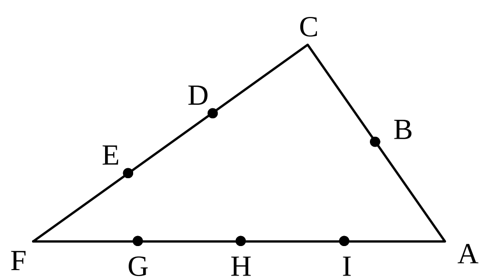
How many distinguishable rearrangements with no fixed letters does $T\,E\,R\,E\,S\,A$ have? (The two Es are considered identical.)

Problem 2/43. Find five different sets of three positive integers $\{k, m, n\}$, such that $k < m < n$ and

$$\frac{1}{k} + \frac{1}{m} + \frac{1}{n} = \frac{19}{84}.$$

Problem 3/43. Suppose $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients and suppose $(p(x))^2$ is a polynomial all of whose coefficients are non-negative. Is it necessarily true that all the coefficients of $p(x)$ must be non-negative? Justify your answer.

Problem 4/43. As shown in the figure on the right, in $\triangle ACF$, B is the midpoint of $\overline{AC}$, D and E divide side $\overline{CF}$ into three equal parts, while G, H and I divide side $\overline{FA}$ into four equal parts.



Seventeen segments are drawn to connect these six points to one another and to the opposite vertices of the triangle. Determine the points interior to $\triangle ACF$ at which three or more of these line segments intersect one another.

Problem 5/43. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, spaced 14 units apart, measured along the straight line connecting them. If the radii of the circles are 18 and 25 units, what is the radius of the sphere?