International Mathematical Talent Search – Round 28

Problem 1/28. For what integers b and c is $x = \sqrt{19} + \sqrt{98}$ a root of the equation $x^4 + bx^2 + c = 0$?

Problem 2/28. The sides of a triangle are of length $a, b,$ and c, where $a, b,$ and c are integers, $a > b$, and the angle opposite to c measures 60°. Prove that a must be a composite number.

Problem 3/28. Determine, with a mathematical proof, the value of $\lfloor x \rfloor$; i.e., the greatest integer less than or equal to x, where

$$x = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{1,000,000}}.$$

Problem 4/28. Let n be a positive integer and assume that for each integer $k,\ 1 \le k \le n,$ we have two disks numbered k. It is desired to arrange the $2n$ disks in a row so that for each $k,\ 1 \le k \le n,$ there are k disks between the two disks that are numbered k. Prove that

(i) if $n = 6$, then no such arrangement is possible;

(ii) if $n = 7$, then it is possible to arrange the disks as desired.

Problem 5/28. Let S be the set of all points of a unit cube (i.e., a cube each of whose edges is of length 1) that are at least as far from any of the vertices of the cube as from the center of the cube. Determine the shape and volume of S.