International Mathematical Talent Search – Round 16

Problem 1/16. Prove that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

Problem 2/16. For a positive integer n, let $P(n)$ be the product of the nonzero base 10 digits of n. Call n "prodigitious" if $P(n)$ divides n. Show that one can not have a sequence of fourteen consecutive positive integers that are all prodigitious.

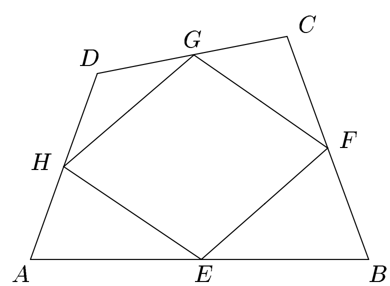
Problem 3/16. Disks numbered 1 through n are placed in a row of squares, with one square left empty. A move consists of picking up one of the disks and moving it into the empty square, with the aim to rearrange the disks in the smallest number of moves so that disk 1 is in square 1, disk 2 is in square 2, and so on until disk n is in square n and the last square is empty. For example, if the initial arrangement is

3	2	1	6	5	4		9	8	7	12	11	10

then it takes at least 14 moves; i.e., we could move the disks into the empty square in the following order: $7, 10, 3, 1, 3, 6, 4, 6, 9, 8, 9, 12, 11, 12$.

What initial arrangement requires the largest number of moves if $n = 1995$? Specify the number of moves required.

Problem 4/16. Let $ABCD$ be an arbitrary convex quadrilateral, with E, F, G, H the midpoints of its sides, as shown in the figure on the right. Prove that one can piece together triangles AEH, BEF, CFG, DGH to form a parallelogram congruent to parallelogram $EFGH$.



Problem 5/16. An equiangular octagon $ABCDEFGH$ has sides of length $2, 2\sqrt{2}, 4, 4\sqrt{2}, 6, 7, 7, 8$. Given that $AB = 8$, find the length of EF.