COMC 2021 Unofficial Solutions

November 4, 2021

Part A

1. **A1** Let x be a real number such that $(x-2)(x+2) = 2021$. Determine the value of $(x-1)(x+1)$.

   Solution: Expanding $(x-2)(x+2) = 2021$ yields $x^2 - 4 = 2021$. Hence, $x^2 = 2025$. Therefore, $(x-1)(x+1) = x^2 - 1 = 2025 - 1 = 2024$.

   Answer: $\boxed{2024}$.

2. **A2** Julia had eaten a box of candies in 4 days. On the first day she ate $\frac{1}{5}$ of the total number of candies. On the second day she ate half of what was left after the first day. On the third day she ate half of what was left after the second day. What portion of the candies initially contained in the box did she eat on the fourth day? Express your answer as an irreducible fraction.

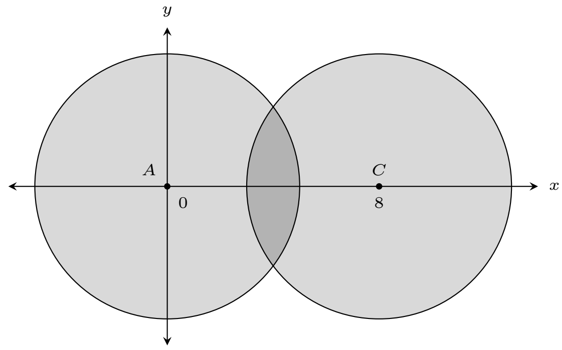
   Solution:

day #	1	2	3	4
eat of the total	$\frac{1}{5}$	$\frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$	$\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$	$\frac{1}{5}$
left of the total	$1 - \frac{1}{5} = \frac{4}{5}$	$\frac{4}{5} - \frac{2}{5} = \frac{2}{5}$	$\frac{2}{5} - \frac{1}{5} = \frac{1}{5}$	0

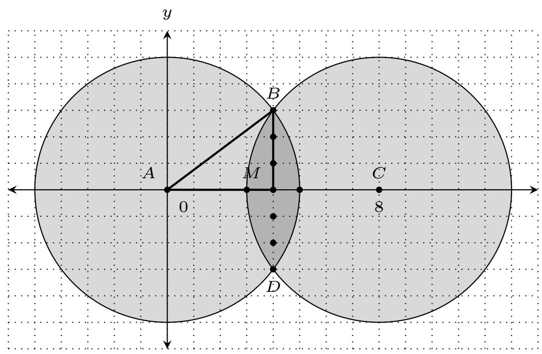
   Answer: $\boxed{1/5}$.

3. **A3**

   Two circles, each of radius 5 units long, are drawn in the coordinate plane such that their centres A and C have coordinates $(0, 0)$ and $(8, 0)$ respectively. How many points where both coordinates are integers are within the intersection of these circles (including its boundary)?



Solution: Let M be the midpoint of AC and the circumferences of the circles intersect at points B and D. Then AM=4 units and AMB is a right triangle with AB=5.



Nowing the $3-4-5$ Pythagorean triple, MB=3 units. Therefore, there are 7 lattice points on BD (including the endpoints). There are two more points on the boundary of the intersection: they have coordinates $(3, 0)$ and $(5, 0)$. The total is 9 lattice points.

Answer: $\boxed{9}$.

4. **A4** Marija travels to school by a combination of walking and skateboarding. She can get there in 38 minutes if she walks for 25 minutes and skateboards for 13, or in 31 minutes if she walks for 11 and skateboards for 20. How long (in minutes) would it take her to walk to school?

   Solution: Denoting her walking speed by w, skateboarding speed by s, and distance to school by D, we have

$$D = 25w + 13s, \qquad \text{and } D = 11w + 20s.$$

   Subtracting these gives $7s = 14w$, and so $D = 51w$. Hence, it will take 51 minutes.

   Alternatively: decreasing the skateboarding time by 7 minutes increases the total time taken by 7 minutes. So decreasing the skateboarding time by 13 minutes will increase the total time take by 13 minutes, from 38 to 51.

   Answer: $\boxed{51}$.

Part B

5. **B1** A bag contains two regularly shaped (cubic) dice which are identical in size. One die has the number 2 on every side. The other die has the numbers 2 on three sides and number 4 on each side opposite to one that has number 2. You pick up a die and look at one side of it, observing the number 2. What is the probability the opposite side of the die has the number 2 as well?

   Express your answer as an irreducible fraction.

   Solution: There are 9 different sides we could be looking at: either side of the first dice, or 3 sides of the second dice. In 6 of those possibilities (i.e. for each side of the first dice), the other side is also a 2, so the probability is $\frac{6}{9} = \frac{2}{3}$.

   Answer: $\boxed{2/3}$.

6. **B2** When the product

$$(2021x^{2021} + 2020x^{2020} + \cdots + 3x^3 + 2x^2 + x)(x^{2021} - x^{2020} + \cdots + x^3 - x^2 + x - 1)$$

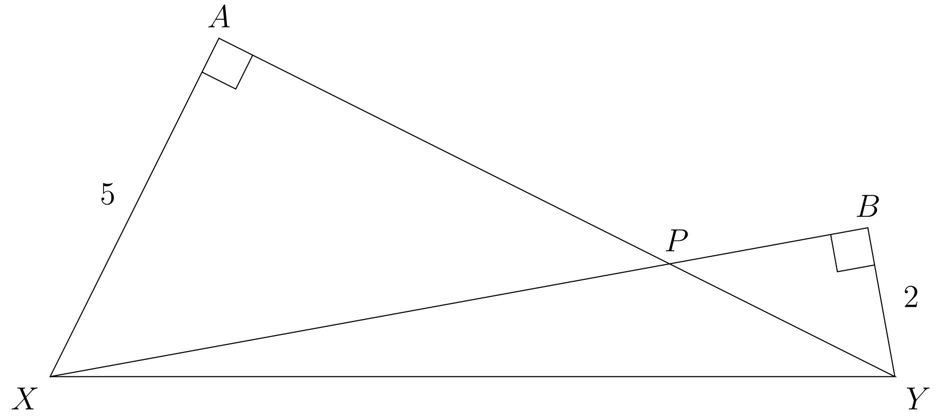
   is expanded and simplified, what is the coefficient of x^{2021}?

   Solution: Expanding terms that yield x^{2021} we get

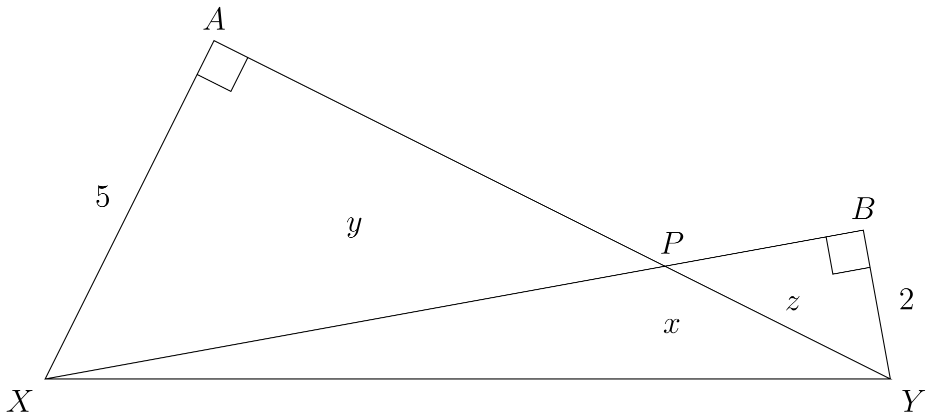
$$\begin{aligned}
(2021x^{2021})(-1) + (2020x^{2020})(x) + (2019x^{2019})(-x^2) + \cdots = \\
&= -2021x^{2021} + 2020x^{2021} - 2019x^{2021} + \cdots \\
&= (-2021 + [2020 - 2019] + [2018 - 2017] + \cdots + [4 - 3] + [2 - 1])x^{2021} \\
&= (-2021 + 1010 \times 1)x^{2021} \\
&= -1011x^{2021}
\end{aligned}$$

   $\therefore$ The coefficient of x^{2021} is -1011.

7. **B3** Two right triangles ΔAXY and ΔBXY have a common hypotenuse XY and side lengths (in units) $AX = 5$, $AY = 10$, and $BY = 2$. Sides AY and BX intersect at P. Determine the area (in square units) of ΔPXY.



Solution:



From AXY we find $XY^2 = 125$. Then from BXY, $BX = 11$.

Labelling the area of the three triangles x, y and z, as in the diagram, we get

$$x + y = \frac{1}{2}(5)(10) = 25$$
$$x + z = \frac{1}{2}(2)(11) = 11$$

As ΔPAX and ΔPBY are right angled and $\angle APX = \angle BPY$, the triangles are similar. Then, since $\frac{BY}{AX} = \frac{2}{5}$ we can deduce that $z = \left(\frac{2}{5}\right)^2 y = \frac{4}{25}y$. Substituting into the system of equations and solving gives the desired area is $\frac{25}{3}$ square units.

Answer: $\boxed{25/3}$.

8. **B4** The equation $\sin x = \frac{x}{2021\pi}$ has exactly n solutions. Find n.

Solution: Notice that $x = 0$ is a solution. As well, both functions are odd, so due to the symmetry if x^* is a solution then $-x^*$ is also a solution.

Since $-1 \le \sin x \le 1$, all solutions are in the interval for which $-1 \le \frac{x}{2021\pi} \le 1$, that is $[-2021\pi, 2021\pi]$.

The function $y = \sin x$ is 2π-periodic. There are 1010 periods and a half of a period on the segment $(0, 2021\pi]$. There are 2 intersections of the line $y = \frac{x}{2021\pi}$ and the sine on each period except the first one $(0, 2\pi]$, where we have only one intersection. There are also two intersections on the last half of the period. Consequently, there are $1 + 2020 = 2021$ solutions on $(0, 2021\pi]$. Similarly, there 2021 solutions on $[-2021\pi, 0)$. In total, there are $n = 1 + 2 \times 2021 = 4043$ solutions.

Answer: $\boxed{4043}$.

Part C

9. **C1**

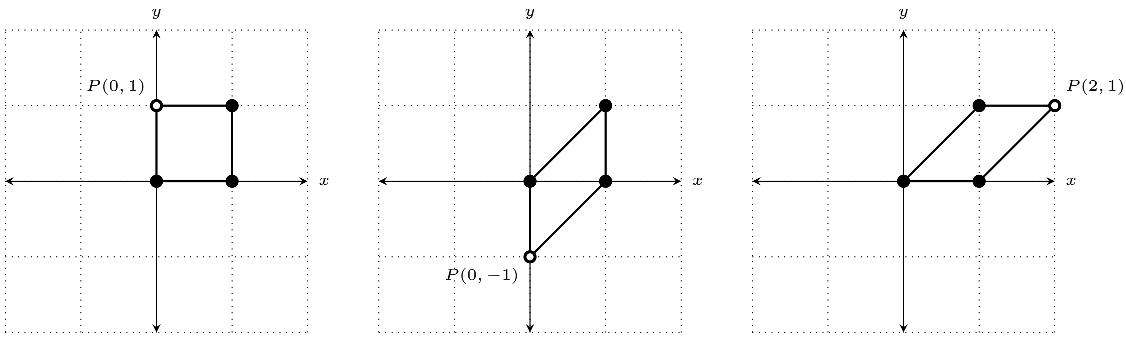
(a) Determine all points $P(x, y)$ such that $(0,0)$, $(1,1)$, $(1,0)$ and P are vertices of a parallelogram.

(b) Two parallel lines intersect the (horizontal) parabola $x = y^2$ at four distinct points: $(0,0)$, $(1,1)$, $(9,3)$ and Q. Determine all possible coordinates of the point Q.

(c) Two parallel lines intersect the parabola $x = y^2$ at four distinct points: $(0,0)$, $(1,1)$, (a^2, a) and V. Here $a \neq 0, \pm 1$ is a real number. Determine all possible coordinates of the point V. The answer should be expresed in term of a.

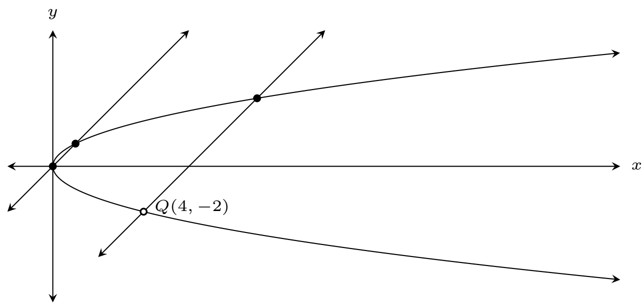
Solution:

(a) There are three possibilities for P: $(0,1)$, $(0,-1)$, $(2,1)$.

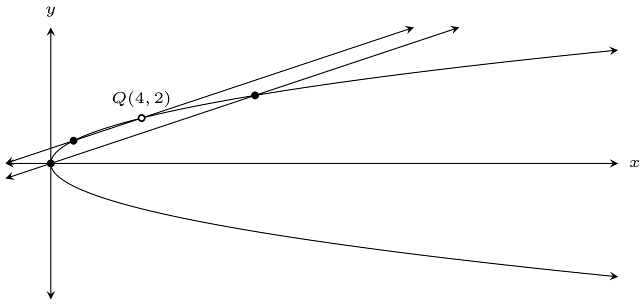


(b) There are three possible pairs of parallel lines - the line through $(0,0)$ can go through $(1,1)$, or $(9,3)$ or Q.

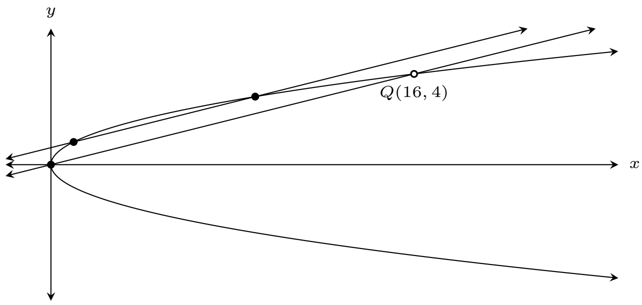
- The line through $(0,0)$ and $(1,1)$ is parallel to $y = x - 6$ through $(9,3)$. The solutions to $y+6 = y^2$ give the y coordinates of the intersection points as $x = y + 6$ and $x = y^2$. Hence at $y = 3$ or $y = -2$, so the y coordinate of Q here is -2 and then $x = 4$.



- The line through $(0,0)$ and $(9,3)$ is parallel to $y = \frac{x+2}{3}$ through $(1,1)$. The solutions to $3y - 2 = y^2$, are $y = 1$ or $y = 2$, so the y coordinate of Q here is 2 and $x = 4$.



- The line through $(1,1)$ and $(9,3)$ is parallel to $y=\frac{x}{4}$ through $(0,0)$. The non-zero solution to $4y=y^2$ is $y=4$, so the y coordinate of Q here is 4 and $x=16$



Thus, Q could be at $(4,-2)$, $(4,2)$, $(16,4)$.

(c) There are three possible pairs of parallel lines - the line through $(0,0)$ can go through $(1,1)$, or (a^2,a) or V.

- The line through $(0,0)$ and $(1,1)$ is parallel to $y=x+a-a^2$ through (a^2,a). The solutions to $y+a(a-1)=y^2$ give the y coordinates of the intersection points, hence at $y=a$ or $y=1-a$, so the y coordinate of V here is $1-a$ and then $x=(1-a)^2$.
- The line through $(0,0)$ and (a^2,a) is parallel to $y=\frac{x+a-1}{a}$ through $(1,1)$. The solutions to $ay+1-a=y^2$, are $y=1$ or $y=a-1$, so the y coordinate of V here is $a-1$ and $x=(a-1)^2$.
- The line through $(1,1)$ and (a^2,a) is parallel to $y=\frac{x}{a+1}$ through $(0,0)$. The non-zero solution to $(a+1)y=y^2$ is $y=a+1$, so the y coordinate of V here is $a+1$ and $x=(a+1)^2$.

Thus, V could be at $((a-1)^2,\pm(a-1)$ and $((a+1)^2,a+1)$.

10. **C2** Let $m,n\geq 2$ be positive integers. Each entry of an $m\times n$ grid contains a real number in the range $[-1,1]$, i.e. between -1 and 1 inclusively. The grid also has the property that the sum of the four entries in every 2×2 *subgrid* is equal to 0. (A 2×2 subgrid is the intersection of two adjacent rows and two adjacent columns of the original grid.)

Let S be the sum of all of the entries in the grid.

(a) Suppose $m=6$ and $n=6$. Explain why $S=0$.

(b) Suppose $m = 3$ and $n = 3$. If the elements of the grid are

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline \end{array},$$

show that $S + e = a + i = c + g$.

(c) Suppose $m = 7$ and $n = 7$. Determine the maximum possible value of S.

Solution:

(a) A 6×6 grid can be partitioned into 2×2 subgrids, because there is an even number of squares along each side of the grid. Since the sum of the entries of each of the subgrids is 0, the sum of all of the entries of the grid is 0. Therefore, $S = 0$.

(b) We have $S = a + b + c + d + e + f + g + h + i$ and the sum of the four entries in every 2×2 subgrid is equal to 0. Thus,

$$S + e = (b + c + e + f) + (d + e + g + h) + a + i = 0 + 0 + a + i = a + i.$$

$$S + e = (a + b + d + e) + (e + f + h + i) + c + g = 0 + 0 + c + g = c + g.$$

(c) The answer is 7.

The sum of all of the entries of the grid can be determined simpler with the following fact.

Fact: In the following entries of a 3×3 grid,

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline 0 & h & i \\ \hline \end{array}$$

the sum of the entries is $S = c - e$. The proof follows from (b), setting $g = 0$.

We can partition the 7×7 grid as follows:

$\cdots$	$\cdots$	$\cdots$	$\cdots$	a_4	b_4	c_4
$\cdots$	$\cdots$	$\cdots$	$\cdots$	d_4	e_4	f_4
$\cdots$	$\cdots$	a_3	b_3	c_3	g_4	h_4
$\cdots$	$\cdots$	d_3	e_3	f_3	$\cdots$	$\cdots$
a_2	b_2	c_2	g_3	h_3	$\cdots$	$\cdots$
d_2	e_2	f_2	$\cdots$	$\cdots$	$\cdots$	$\cdots$
c_1	g_2	h_2	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The entries not labeled can be broken down into 2×2 grid, and hence has sum 0. Hence, from this observation, and the fact we used above, the sum of all of the entries of the 7×7 grid is

$$c_1 - e_2 + c_2 - e_3 + c_3 - e_4 + c_4.$$

Since each entry lies between -1 and 1 inclusively, this sum is at most 1. Hence, the sum of all the entries is at most 7. This maximum sum can be obtained as follows:

1	0	1	0	1	0	1
0	-1	0	-1	0	-1	0
1	0	1	0	1	0	1
0	-1	0	-1	0	-1	0
1	0	1	0	1	0	1
0	-1	0	-1	0	-1	0
1	0	1	0	1	0	1

or similarly to (b):

1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1

11. **C3** Xintong plays a game of turning one six-digit number into another. The numbers can have leading zeros, but cannot go over 6 digits or below 0. He can only make the following moves, any number of times in any order:

- R: rotate the last digit to the start, for example, $092347 \to 709234$, or
- A: add 1001 to the number, for example, $709234 \to 710235$, or
- S: subtract 1001 from the number, for example, $709234 \to 708233$.

(a) Show that it is possible to turn 202122 into 313233.

(b) Show that turning 999999 into 000000 can be done in eight moves.

(c) Show that any multiple of 11 remains a multiple of 11 after any sequence of moves.

(d) Show that it is impossible to turn 112233 into 000000.

Solution:

(a) We need to add one to every digit, so any combination of three additions to distinct digits and six rotations will work. For example, it can be reached by $A, R, A, R, A, R, R, R, R$.

(b) This can be done through moves: S, R, A, R, R, A, R, S.

(c) This can be done similarly to the next part, or by using "If n has digits $a_1a_2a_3a_4a_5a_6$, then n is divisible by 11 if and only if $a_1 - a_2 + a_3 - a_4 + a_5 - a_6$ is." Adding or subtracting 1001 does not change the sum, and rotating makes it negative, so divisibility by 11 is preserved.

(d) We can show that (non-)divisibility by 7, or 11, or 13 is an invariant. If n is not divisible by 7 or 11 or 13, then the same is true for $n \pm 1001$, since $7 \cdot 11 \cdot 13 = 1001$. Letting k denote the last digit of n, the rotation move replaces n by $\frac{n-k}{10} + 10^5k$. Multiplying by 10 and noting that $10^6 \equiv 1$ modulo each of $7, 11, 13$, we get that the rotated number is congruent to $\frac{n}{10}$ for each one, or equivalently, the new number is $5n$ mod 7, $-n$ mod 11, and $4n$ mod 13.Thus, any number that is not a multiple of 7 or 11 or 13 cannot be turned into one. Note that $112233 \equiv 2123 \equiv 121 \pmod 7$, so it cannot turn into 000000.

12. **C4** We call (F, c) a *good pair* if the following three conditions are satisfied:

(1) $F(x) = a_0 + a_1x + \cdots a_mx^m$, $(m \geq 1)$ is a non-constant polynomial with integer coefficients.

(2) c is a real number that is not an integer.

(3) $F(c)$ is an integer.

For example, both $(6x, \frac{1}{3})$ and $(1+x^3, 5^{1/3})$ are good pairs, but none of the following pairs $(6x, \frac{1}{4})$, $(6x, 2)$, $(\frac{x}{6}, \frac{1}{3})$, $(\frac{x^2}{6}, 6)$ is good.

(a) Let $c = \frac{1}{2}$. Give an example of F such that (F, c) is a good pair but $(F, c+1)$ is not.

(b) Let $c = \sqrt{2}$. Give an example of F such that both (F, c) and $(F, c+1)$ are good pairs.

(c) Show that for any good pair (F, c), if c is rational then there exists infinitely many non-zero integers n such that $(F, c+n)$ is also a good pair.

(d) Show that if $(F, c+n)$ is a good pair for every integer n, then c is rational.

Solution

(a) $F(x) = 2x^3 + x^2 + x$; $F(\frac{1}{2}) = 1$ and $F(\frac{3}{2}) = \frac{21}{2}$.

(b) $F(x) = (x^2 - 2)((x-1)^2 - 2))$; $F(\sqrt{2}) = F(\sqrt{2}+1) = 0$.

(c) Let $c = \frac{p}{q}$ and let $F(x) = \sum_{i=0}^m a_ix^i$. Then, $F(c+n) - F(c) = \sum_{i=0}^m a_i\left((c+n)^i - c^i\right)$. For all $1 \leq i \leq m$ we have

$$(c+n)^i - c^i = \sum_{j=0}^{i} \binom{i}{j} c^j n^{i-j} - c^i = n\sum_{j=0}^{i-1} \binom{i}{j} c^j n^{i-1-j} = n\sum_{j=0}^{i-1} \binom{i}{j} \left(\frac{p}{q}\right)^j n^{i-1-j}.$$

If now we choose $n = q^k$, $k \geq m-1$ then each of these differences is an integer. So there are infinitely many choices for n.

(d) Assume otherwise; that is, there is some irrational number c such that $F(c+n)$ is an integer for every integer n. Take such a polynomial F of minimal degree. It is clear that F is not linear. But now we can take a polynomial $G(x) = F(x+1) - F(x)$, and now $G(x)$ has degree one lower than F and is also an integer on $c+n$ for all integers n, which is a contradiction.

Note that if $c = \frac{p}{q}$ is rational then $F(x) = qx$ makes the required good pairs.