Canadian Open Mathematics Challenge 2020

Official Solutions

A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

The COMC has three sections:

A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

COMC exams from other years, with or without the solutions included, are free to download online. Please visit http://comc.math.ca/2020/practice.html

 January 26, 2021

Section A – 4 marks each

A1. At a party, if each kid took one apple from the fruit bucket then 7 apples would still remain in the bucket. However, if each kid had an appetite for two apples, the supply would be 16 apples short. How many kids were at the party?

Solution: Let a and k be the number of apples and kids respectively. We have $a - k = 7$ and $a - 2k = -16$. Then $a = 7 + k$ and $k = 7 + 16 = 23$.

Answer: $\boxed{23}$

A2. It is possible to create 24 distinct four digit numbers, where each number uses each of the digits 1, 2, 3, and 4 exactly once. How many of these are divisible by 4?

Solution: The number is divisible by 4 if and only if the last two digits are 12, 32, or 24. There are two numbers ending in each of these cases, so 6 of these numbers are divisible by 4.

Answer: $\boxed{6}$

Alternative solution: One could list the numbers explicitly:

3412, 4312, 1432, 4132, 1324, 3124.

A3. One person can make n objects in an hour. Ann begins at 10:00 AM, Bob begins at 10:20 AM, then Cody and Deb begin at 10:40 AM. Working at the same constant speed, by 11:00 AM, they have made 28 objects. What is n?

Solution: At 11 am, Ann has worked for 1 hour, Bob has worked for $\frac{2}{3}$ of an hour, and both Cody and Deb have worked for $\frac{1}{3}$ of an hour. Together, they have worked for $1 + \frac{2}{3} + \frac{2}{3} = \frac{7}{3}$ hours. Thus, $n = 28 \div \frac{7}{3} = 12$.

Answer: $\boxed{12}$

A4. If a, b are the roots of the polynomial $x^2 + x - 2020 = 0$, find $a^2 - b$.

Solution:

Since a and b satisfy the equation, we have $a^2 + a - 2020 = 0$ and $a + b = -1$. Then $a^2 - b = -a + 2020 - b = -(a + b) + 2020 = 1 + 2020 = 2021$.

Alternative solution: Solve the equation to get $a = \frac{-1+\sqrt{8081}}{2}$ and $b = \frac{-1-\sqrt{8081}}{2}$. Then

$$a^2 - b = \frac{1 - 2\sqrt{8081} + 8081}{4} - \frac{-1 - \sqrt{8081}}{2} = \frac{8084}{4} = 2021.$$

For another choice $a = \frac{-1-\sqrt{8081}}{2}$ and $b = \frac{-1+\sqrt{8081}}{2}$ we get the same answer

$$a^2 - b = \frac{1 + 2\sqrt{8081} + 8081}{4} - \frac{-1 + \sqrt{8081}}{2} = \frac{8084}{4} = 2021.$$

Answer: $\boxed{2021}$

Section B – 6 marks each

B1. The set S contains 6 distinct positive integers. The average of the two smallest numbers is 5 and the average of the two largest is 22. What is the greatest possible average of all the numbers in the set S?

Solution: Let $S = \{s_1, \ldots, s_6\}$, where $s_1 < s_2 < \cdots < s_6$. Then $s_1 + s_2 = 10$ and $s_5 + s_6 = 44$. Since s_5 and s_6 are distinct, we have $s_5 \le 21$. But then $s_4 \le s_5 - 1 \le 20$ and $s_3 \le s_4 - 1 \le 19$. Thus

$$s_1 + \cdots + s_6 = 10 + s_3 + s_4 + 44 \le 10 + 19 + 20 + 44 = 93$$

The greatest possible average is therefore $\frac{93}{6} = \frac{31}{2} = \boxed{15.5}$.

Note that this is indeed obtained with $S = \{4, 6, 19, 20, 21, 23\}$.

Answer: $\boxed{15.5}$

B2. Alice places a coin, heads up, on a table then turns off the light and leaves the room. Bill enters the room with 2 coins and flips them onto the table and leaves. Carl enters the room, in the dark, and removes a coin at random. Alice reenters the room, turns on the light and notices that both coins are heads. What is the probability that the coin Carl removed was also heads?

Solution :

Since we started with one coin heads up, there are 4 possible configurations before Carl enters the room: HHH, HHT, HTH, HTT. Since there are two heads when Carl left, the configuration HTT was impossible. For the remaining configurations:

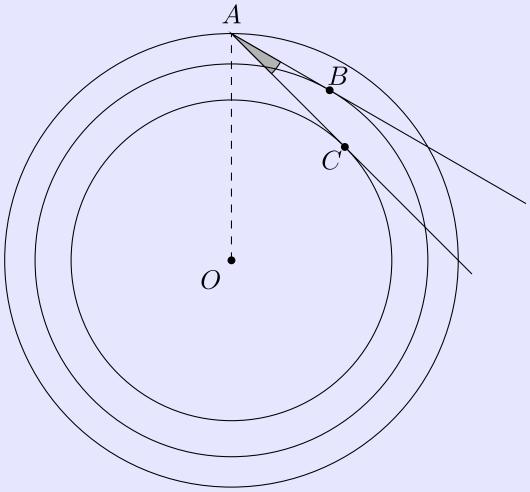
Case 1: HHH There are 3 ways Carl could leave 2 heads, they all involve him removing a head.

Case 2: HHT or HTH There is only one way for each of these that would leave 2 heads, both involve removing a tail.

Therefore, the probability that Carl removes a head is $\frac{3}{5}$.

Answer: $\boxed{\frac{3}{5}}$

B3. Three circles with the same centre O and areas 2π, 3π and 4π are drawn. From a point A on the largest circle, tangent lines are drawn to points B on the middle circle and C on the smallest circle. If B, C are on the same side of OA, find the exact value of $\angle BAC$.



Solution: Draw segments OB and OC. Note that ABO and ACO are right angles. The line segment OB is the radius of a circle with area 3π, so it has length $OB = \sqrt{3}$. Similarly, the line segment OC is the radius of a circle with area 2π, so it has length $OC = \sqrt{2}$, and OA has length 2.

From ABO, $\sin \angle BAO = \frac{\sqrt{3}}{2}$, so $\angle BAO = 60°$.

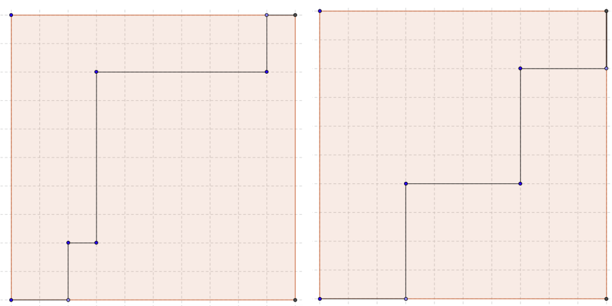
From ACO, $\sin \angle CAO = \frac{\sqrt{2}}{2}$ so $\angle CAO = 45°$.

Then $\angle BAC = \angle BAO - \angle CAO = 60° - 45° = 15°$.

Answer: $\boxed{15°}$

B4. An ant walks from the bottom left corner of a 10×10 square grid to the diagonally-opposite corner, always walking along grid lines and taking as short a route as possible. Let $N(k)$ be the number of different paths that ant could follow if it makes exactly k turns. Find $N(6) - N(5)$.

Solution: These paths are parameterized by their starting direction (north or east) and the lengths of all northward or eastward segments $\{a_j, j = 1, .., k+1\}$, which must sum to 10 in each direction. The number of segments is one more than the number of turns, so for $k = 5$ we have $a_1 + a_3 + a_5 = 10$ and $a_2 + a_4 + a_6 = 10$ and for $k = 6$ we have $a_1 + a_3 + a_5 + a_7 = 10$ and $a_2 + a_4 + a_6 = 10$. Sample paths starting east with 6 and 5 turns respectively are shown below:



Let P_ℓ be the number of positive integer solutions to the equation $b_1 + b_2 + \cdots + b_\ell = 10$.

It is easy to see that $P_\ell = \binom{9}{\ell-1}$ because there are $10 - 1 = 9$ spaces in a row between 10 objects, and we pick $\ell - 1$ of them to place a delimiter; the number of objects between two adjacent delimiters are b_i respectively.

We need $P_3 = \binom{9}{2} = 36$ and $P_4 = \binom{9}{3} = 84$.

Then the desired number of paths is $N(5) = 2P_3P_3 = 2592$ and $N(6) = 2P_3P_4 = 6048$.

The factor of 2 accounts for the option to begin either to the north or east.

Thus, $N(6) - N(5) = 2P_3(P_4 - P_3) = 2 \times 36 \times (84 - 36) = \boxed{3456}$.

Alternative solution: After fixing the initial direction, the set of rows and set of columns we 'turn onto' uniquely determine the walk. For example, if the first step is horizontal, then for k even the last step must be horizontal, which means that the we are forced to choose the last row as one of the rows that we 'turn into'. Similarly, when k is odd, we are forced to turn into the last column. Thus, there are $\binom{n-1}{\text{floor}((k-1)/2)}$ possible sets of rows and $\binom{n-1}{\text{ceiling}((k-1)/2)}$ sets of columns.

Then we double the result to account for the other initial direction.

In our case $n = 10$ and $k = 6$ or $k = 5$, so

$N(6) - N(5) = 2\binom{9}{2}\binom{9}{3} - 2\binom{9}{2}\binom{9}{2} = 2 \times 36 \times (84 - 36) = \boxed{3456}$.

Answer: $\boxed{3456}$

Section C – 10 marks each

C1. Find the areas of the three polygons described by the following conditions (a), (b) and (c), respectively.

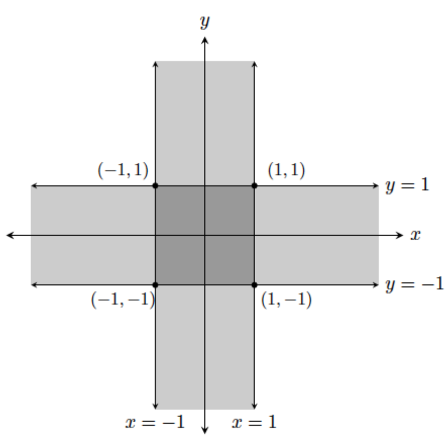
(a) the system of inequalities $|x| \leq 1$ and $|y| \leq 1$

(b) the inequality $|x| + |y| \leq 10$

(c) the inequality $|x| + |y| + |x + y| \leq 2020$

Solution for C1(a):
Here we have all points (x, y) such that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

This is a square with vertices $(1, 1), (1, -1), (-1, 1), (-1, -1)$. The area is $\boxed{4}$.

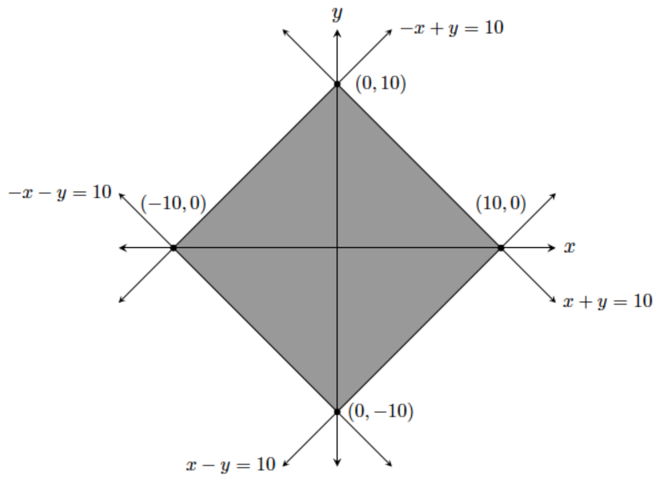


Solution for C1(b) Consider the first quadrant $x \geq 0, y \geq 0$. Then $|x| = x, |y| = y$ and we have $x + y \leq 10$. So, we have all points (x, y) inside the triangle with vertices $(0, 0), (10, 0), (0, 10)$.

Consider the second quadrant $x \leq 0, y \geq 0$. Then $|x| = -x, |y| = y$ and we have $-x + y \leq 10$. So, we have all points (x, y) inside the triangle with vertices $(0, 0), (-10, 0), (0, 10)$.

Consider the third quadrant $x \leq 0, y \leq 0$. Then $|x| = -x, |y| = -y$ and we have $-x - y \leq 10$, equivalently, $x + y \geq -10$. So, we have all points (x, y) inside the triangle with vertices $(0, 0), (-10, 0), (0, -10)$.

Consider the fourth quadrant $x \leq 0, y \leq 0$. Then $|x| = x, |y| = -y$ and we have $x - y \leq 10$. So, we have all points (x, y) inside the triangle with vertices $(0, 0), (10, 0), (0, -10)$.



The area of each of the four right triangles is $(10 \times 10)/2 = 50$. They form the square with vertices $(\pm 10, 0), (0, \pm 10)$. The total area of the square is $\boxed{200}$.

C1(b) Alternative solution:
Consider the first quadrant $x \geq 0, y \geq 0$. Then $|x| = x, |y| = y$ and we have $x + y \leq 10$. So, we have all points (x, y) inside the triangle with vertices $(0,0), (10,0), (0,10)$. By symmetry, the entire figure is the rotated square centered at the origin. Its side length is $\|(10,0) - (0,10)\| = \sqrt{(10^2 + 10^2)} = \sqrt{200}$. Thus, the square has area $\sqrt{200}^2 = \boxed{200}$.

Solution for C1(c):

Consider first the region bounded by $|x| + |x + y| + |y| = 2$.

We work out the cases:

1) If $x \geq 0, y \geq 0$ then $x + x + y + y \leq 2$ so $x + y \leq 1$. This gives a triangle with vertices $(0,0), (0,1), (1,0)$. The area is 0.5.

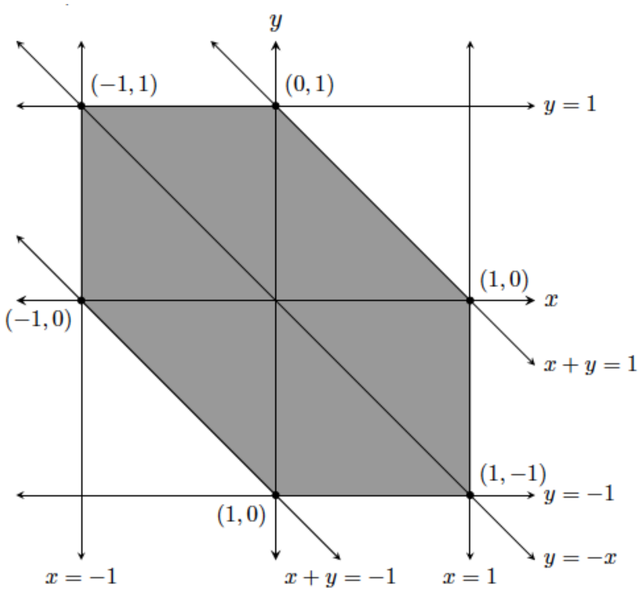
2) If $x \leq 0, y \geq 0$ and $x \geq -y$ then$-x + x + y + y \leq 2$ so $y \leq 1$. This gives a triangle with vertices $(0,0), (0,1), (-1,1)$. The area is 0.5.

3) If $x \leq 0, y \geq 0$ and $x \leq -y$ then$-x - x - y + y \leq 2$ so $x \geq 1$. This gives a triangle with vertices $(0,0), (-1,0), (-1,1)$. The area is 0.5.

4) If $x \leq 0, y \leq 0$ then $-x - x - y - y \leq 2$ so $x + y \geq -1$ This gives a triangle with vertices $(0,0), (0,-1), (-1,0)$. The area is 0.5.

5) If $x \geq 0, y \leq 0$ and $x \geq -y$ then $x + x + y - y \leq 2$ so $x \leq 1$. This gives a triangle with vertices $(0,0), (1,0), (1,-1)$. The area is 0.5.

6) If $x \geq 0, y \leq 0$ and $x \leq -y$ then $x - x - y - y \leq 2$ so $y \geq -1$. This gives a triangle with vertices $(0,0), (0,-1), (1,-1)$. The area is 0.5.



Consequently, the six triangles form a hexagon with vertices $(\pm 1, 0), (0, \pm 1)$ and $(-1, 1), (1, -1)$, which has area 3. Scaling by a linear factor of 1010 gives an area of $3(1010)^2 = 3060300$.

Alternatively, without re-scaling, area of each triangle will be $1010^2/2 = 505 \times 1010 = 510050$. The six triangles give $510050 \times 6 = \boxed{3060300}$.

Alternative solution C1(c):

This area is fully contained within the square $\begin{cases} |x| \leq 1010 \\ |y| \leq 1010 \end{cases}$.

What parts of the square ($|x| \leq 1010$ & $|y| \leq 1010$) are in $|x| + |y| + |x + y| \leq 2020$?
The complement is $|x| \leq 1010, |y| \leq 1010$ and $|x + y| > 1010$. That is, we subtract 2 triangles of area $\frac{1}{2}(1010)^2$ (a total of 1010^2) from a big square of area 2020^2. This is,
$1010^2(4 - 1) = 1020100 \times 3 = \boxed{3060300}$.

C2. An expression like

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

is called a continued fraction.

(a) Write x given above as a reduced fraction of the form $\frac{a}{b}$ where a and b are positive integers.

(b) Write $\dfrac{355}{113}$ as a continued fraction in the form $a + \frac{1}{b+\frac{1}{c}}$, where a, b, c are positive integers.

(c) Let

$$y = 8 + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{8 + \cdots}}}$$

where the process continues indefinitely. Given that y can be written in the form $p + \sqrt{q}$, find the integers p and q.

Solution for (a): Working from the bottom up we get

$$\begin{aligned} 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}} &= 1 + \cfrac{1}{2 + \cfrac{1}{\frac{13}{4}}} \\ &= 1 + \cfrac{1}{2 + \frac{4}{13}} \\ &= 1 + \cfrac{1}{\frac{30}{13}} \\ &= 1 + \frac{13}{30} \\ &= \frac{43}{30} \end{aligned}$$

Solution for (b): Reversing the process from part a) we get

$$\begin{aligned} \frac{355}{113} &= 3 + \frac{16}{113} \\ &= 3 + \cfrac{1}{\frac{113}{16}} \\ &= 3 + \cfrac{1}{7 + \frac{1}{16}} \end{aligned}$$

C2.(c) Let

$$y = 8 + \cfrac{1}{8 + \frac{1}{8 + \frac{1}{8 + \cdots}}}$$

where the process continues indefinitely. Given that y can be written in the form $p + \sqrt{q}$, find the integers p and q.

Solution for (c):

Notice that

$$y = 8 + \frac{1}{y}$$

which yields the quadratic equation

$$y^2 - 8y - 1 = 0$$

with solutions

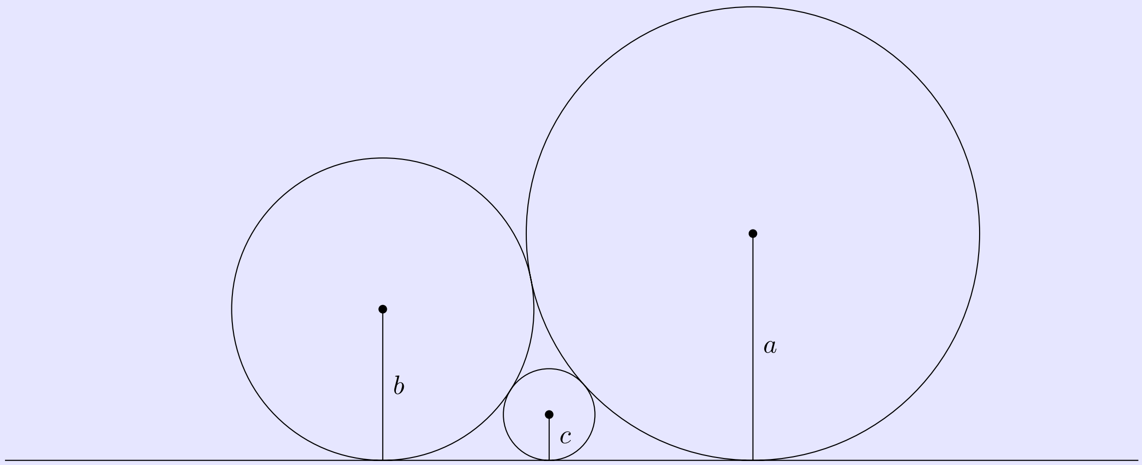
$$y = \frac{8 \pm \sqrt{68}}{2}$$

Since clearly $y \geq 0$, we must have

$$y = 4 + \sqrt{17}.$$

Thus, $p = 4$ and $q = 17$.

C3. Three circles with radii $0 < c < b < a$ are tangent to each other and to a line as shown in the diagram below.



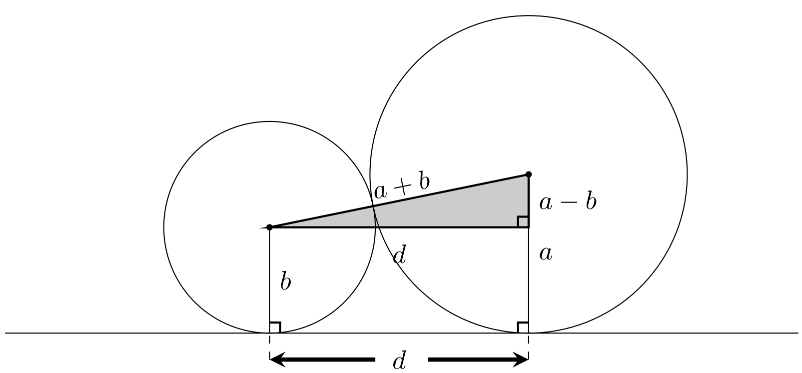
(a) Let radii $a = 16$ and $b = 4$. Find the distance d between the points of tangency of these two circles to the line.

(b) Let radii $a = 16$ and $b = 4$. Find the radius c.

(c) The configuration is called *nice* if a, b, and c are all integers. Among all nice configurations, find the smallest possible value of c.

Solution for (a):

Consider the two circles with radii a and b and let d be the distance between their points of tangency to the line.



Since tangents are perpendicular to radii at the point of tangency, we can create the indicated right triangle which yields

$$d^2 + (a-b)^2 = (a+b)^2 \Rightarrow d = 2\sqrt{ab} = 2\sqrt{16 \times 4} = 16.$$

C3. (b) Let radii $a = 16$ and $b = 4$. Find the radius c.

Solution for (b):

From (a), $d = 2\sqrt{ab}$. Similarly to (a), the distances between the points of tangency of the circles with radii b and c and the circles with radii c and a are $2\sqrt{bc}$ and $2\sqrt{ac}$, respectively. Hence we must have $2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}$, that is $16 = 2\sqrt{16c} + 2\sqrt{4c}$. So, $4/3 = \sqrt{c}$. The answer is $c = 16/9$.

C3. (c) The configuration is called *nice* if a, b, and c are all integers. Among all nice configurations, find the smallest possible value of c.

Solution for (c):

In (b) we noted that

$$2\sqrt{bc} + 2\sqrt{ac} = 2\sqrt{ab}.$$

Solving for c yields

$$\begin{aligned}
\sqrt{bc} + \sqrt{ac} &= \sqrt{ab} \\
(\sqrt{bc} + \sqrt{ac})^2 &= (\sqrt{ab})^2 \\
bc + 2c\sqrt{ab} + ac &= ab \\
c &= \frac{ab}{a + b + 2\sqrt{ab}}.
\end{aligned}$$

Since c is an integer, so must $\sqrt{ab}$ be. Hence a and b must be of the form $a = k\alpha^2$ and $b = k\beta^2$ for positive integers k, α, and β. Take $k = \gcd(a, b)$. Thus

$$\begin{aligned}
c &= \frac{k^2\alpha^2\beta^2}{k\alpha^2 + k\beta^2 + 2k\alpha\beta} \\
&= \frac{k\alpha^2\beta^2}{(\alpha + \beta)^2}
\end{aligned}$$

where $\gcd(\alpha, \beta) = 1$.

Since c is a positive integer, $(\alpha + \beta)^2 \mid k$. We want to make a, b, and c as small as possible, so we can choose $\alpha = 2$ and $\beta = 1$ which gives $9 \mid k$. Thus the minimum solution occurs when $k = 9$ which yields $a = 36$, $b = 9$ and $\boxed{c = 4}$.

C3(c) Alternative solution:

We start with the identity noted in (b)

$$\sqrt{bc} + \sqrt{ac} = \sqrt{ab}, \tag{1}$$

and rewrite it in the form

$$(\sqrt{a} - \sqrt{c})(\sqrt{b} - \sqrt{c}) = c.$$

Since $0 < c < b < a$ the above identity implies

$$\sqrt{a} - \sqrt{c} > \sqrt{c}, \quad \sqrt{b} - \sqrt{c} < \sqrt{c},$$

so that $\sqrt{a} > 2\sqrt{c}$ and $\sqrt{b} < 2\sqrt{c}$ and we conclude that a, b and c must satisfy the following inequalities

$$0 < c < b < 4c < a. \tag{2}$$

Next we rewrite (1) in the form $\sqrt{bc} = \sqrt{ab} - \sqrt{ac}$ and squaring both sides of this identity we obtain

$$bc = ab + ac - 2a\sqrt{bc}.$$

Therefore, if a, b and c are integers then $\sqrt{bc}$ must be a rational number. Then the smallest positive integers c and b that also obey (2) are $\boxed{c = 4}$ and $b = 9$. In this case $a = 36$.

C4. Let $S = \{4, 8, 9, 16, \ldots\}$ be the set of integers of the form m^k for integers $m, k \geq 2$. For a positive integer n, let $f(n)$ denote the number of ways to write n as the sum of (one or more) *distinct* elements of S. For example, $f(5) = 0$ since there are no ways to express 5 in this fashion, and $f(17) = 1$ since $17 = 8 + 9$ is the only way to express 17.

(a) Prove that $f(30) = 0$.

(b) Show that $f(n) \geq 1$ for $n \geq 31$.

(c) Let T be the set of integers for which $f(n) = 3$. Prove that T is finite and non-empty, and find the largest element of T.

Note that all even elements in S are divisible by 4.

Solution for (a):

The elements of S under 30 are $4, 8, 9, 16, 25, 27$. Since $30 \equiv 2 \pmod 4$, if 30 is expressible as a sum, it must use at least two distinct odd numbers (since all even numbers are $0 \pmod 4$). But the smallest two odd numbers on this list are $9, 25$, who sum to 34 which is already too large. Thus 30 is not expressible.

Many students will instead run through some simple casework, i.e. $30 = 27 + 3$ but 3 is too small, $30 = 25 + 5$ but we can't make 5, etc.

Solution for (b):

If $n \equiv 0 \pmod 4$, then write n in binary. Since $1, 2$ do not show up, this is a valid expression for n as a sum of distinct elements of S, hence $f(n) \geq 1$.

If $n \equiv 1 \pmod 4$ and $n \geq 9$, then $n - 9 \equiv 0 \pmod 4$ and $n - 9$ is a nonnegative integer. Write $n - 9$ in binary, and adding back in the 9 is a valid expression since 9 does not appear in the binary expansion.

If $n \equiv 2 \pmod 4$ and $n \geq 34$, then as above we write $n - 34$ in binary, and add back in $34 = 9 + 25$.

Finally, if $n \equiv 3 \pmod 4$ and $n \geq 27$, then write $n - 27$ in binary, and add back in 27.

Combining the above shows that all $n \geq 31$ are expressible as a sum of distinct elements in S, hence $f(n) \geq 1$.

C4. (c) Let T be the set of integers for which $f(n) = 3$. Prove that T is finite and non-empty, and find the largest element of T.

Solution for (c):

We claim that $f(111) = 3$ and $f(n) \geq 4$ for $n \geq 112$, hence T is finite and non-empty with largest element 111.

Note that in part (b) we proved the following stronger statement: all integers $n \geq 31$ are expressible as a sum of distinct elements of the set S', where $S' = \{9, 25, 27\} \cup \{2^k : k \geq 2\}$. If $n \geq 112$, then $n, n-36, n-49, n-81 \geq 31$. Thus by part (b), they are expressible as a sum of distinct elements in S'. Adding back in $0, 36, 49, 81$ will therefore give us valid sums for n that are all distinct, hence $f(n) \geq 4$.

It remains to show that $f(111) = 3$. The odd elements of S less than or equal to 111 are $\{9, 25, 27, 49, 81\}$, and are all equivalent to $1 \pmod 4$ except 27. Since $111 \equiv 3 \pmod 4$, the possibilities for the odd numbers in a valid sum of 111 are:

- All five odd numbers;
- Only 27;
- Three of $9, 25, 49, 81$.

The first case cannot happen since this sum is too big. For the second, it remains to express $111 - 27 = 84$ as a sum from $\{4, 8, 16, 32, 36, 64\}$. If we do not use 36 this is just powers of two, so there is only one way: $111 = 64 + 27 + 16 + 4$. If we use 36, there is again only one way to finish: $111 = 36 + 32 + 27 + 16$. In the last case, if 81 is one of the three numbers, the total sum is at least $81 + 9 + 25 = 115$ which is too large. Thus the only possibility is to use $9 + 25 + 49 = 83$, which leaves 28 to express as a sum from $\{4, 8, 16\}$. These sum to 28, so again we get one solution $111 = 49 + 25 + 16 + 9 + 8 + 4$. Thus there are exactly 3 ways to express 111 as a sum of distinct elements of S, so $\boxed{n = 111}$ is the largest integer with $f(n) = 3$.

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