Canadian Open Mathematics Challenge 2019

Official Solutions

A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

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SOCIETY OF ACTUARIES

The COMC has three sections:

A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

COMC exams from other years, with or without the solutions included, are free to download online. Please visit http://comc.math.ca/2019/practice.html

 January 16, 2020

Section A – 4 marks each

A1. Shawn's password to unlock his phone is four digits long, made up of two 5s and two 3s. How many different possibilities are there for Shawn's password?

Solution: This is the number of ways to choose 2 spots out of 4, thus there are $\binom{4}{2} = 6$ possibilities.

Alternatively one can list all the possibilities: 3355, 3535, 3553, 5335, 5353, 5533.

Answer: $\boxed{6}$.

A2. Triangle ABC has integer side lengths and perimeter 7. Determine all possible lengths of side AB.

Solution: By the triangle inequality, the largest possible length of any side is 3. Moreover, if all side lengths were less than 3, the perimeter would be less than 7. Thus, at least one side has length 3. This leaves the other sides has (2,2) or (1,3). Thus, AB can be 1, 2, or 3.

Answer: $\boxed{1,2,3}$.

A3. If a and b are positive integers such that $a = 0.6\,b$ and $\gcd(a, b) = 7$, find $a + b$.

Solution: $\frac{a}{b} = \frac{6}{10} = \frac{3}{5} = \frac{3\times 7}{5\times 7} = \frac{21}{35}$. Thus, $a = 21$ and $b = 35$, so $a + b = 21 + 35 = 56$.
Answer: $\boxed{56}$.

A4. The equations $|x|^2 - 3|x| + 2 = 0$ and $x^4 - ax^2 + 4 = 0$ have the same roots. Determine the value of a.

Solution:

For $x \geq 0$, equations $|x|^2 - 3|x| + 2 = 0$ becomes $x^2 - 3x + 2 = 0$, which has roots $x = 1$ and $x = 2$. For $x < 0$, the equations $|x|^2 - 3|x| + 2 = 0$ becomes $x^2 + 3x + 2 = 0$, which has roots $x = -1$ and $x = -2$. Thus, $|x|^2 - 3|x| + 2 = 0$ has roots $x = \pm 1, \pm 2$.

A polynomial equation with the same roots has the form: $(x - 1)(x + 1)(x - 2)(x + 2) = 0$. Rewriting the left hand side as $(x^2 - 1)(x^2 - 4) = x^4 - 5x^2 + 4$ and comparing with $x^4 - ax^2 + 4 = 0$, we get $a = 5$.

Alternatively, we can substitute $x = \pm 1, \pm 2$ in $x^4 - ax^2 + 4 = 0$ to get either $1 - a + 4 = 0$ or $16 - 4a + 4 = 0$. In both cases we have $a = 5$.

Answer: $\boxed{5}$.

Section B – 6 marks each

B1. John walks from home to school with a constant speed, and his sister Joan bikes twice as fast. The distance between their home and school is 3 km. If Joan leaves home 15 minutes after John then they arrive to school at the same time. What is the walking speed (in km/h) of John?

Solution 1: Let x (km/h) be the speed of John. Then the speed of Joan is $2x$. The time John spends for walking to school is $3/x$ hours. The time Joan spends for biking to school is $3/(2x)$ hours. Since she leaves home 0.25h after John and arrives to school at the same time, we have the equation:
$3/x = 0.25 + 3/(2x)$. Solving it we find $x = 6$ km/h.

Solution 2: Joan covers twice the distance as John in the same time, so to arrive together Joan must leave when John is halfway. Thus John has walked 1.5 km in 15 minutes, or 1/4 hour. John therefore walks $4 \times 1.5 = 6$ km/hr.

Answer: $\boxed{6}$.

B2. What is the largest integer n such that the quantity

$$\frac{50!}{(5!)^n}$$

is an integer?

Note: Here $k! = 1 \times 2 \times 3 \times \cdots \times k$ is the product of all integers from 1 to k. For example, $4! = 1 \times 2 \times 3 \times 4 = 24$.

Solution : We can calculate $5! = 2^3 \times 3 \times 5$, thus $(5!)^n \mid 50!$ if

$$2^{3n} \mid 50!$$
$$3^n \mid 50! \text{ and}$$
$$5^n \mid 50!$$

(The notation $b \mid a$ means integer number a is divisible by integer number b.)

Observe that among integers from 1 to 50, every fifth integer is divisible by 5, and also 25 and 50 are divisible by 5^2. Therefore, the exponent of the maximal power of 5 that divides 50! is

$$\left\lfloor \frac{50}{5} \right\rfloor + \left\lfloor \frac{50}{5^2} \right\rfloor = 10 + 2 = 12$$

so our desired $n \leq 12$. Doing a similar analysis for the powers of 2 and 3 we get:

$$\left\lfloor \frac{50}{2} \right\rfloor + \left\lfloor \frac{50}{2^2} \right\rfloor + \left\lfloor \frac{50}{2^3} \right\rfloor + \left\lfloor \frac{50}{2^4} \right\rfloor + \left\lfloor \frac{50}{2^5} \right\rfloor = 25 + 12 + 6 + 3 + 1 = 47,$$

so $2^{47} \mid 50!$ and $3n \leq 47$, thus $n \leq 15$.

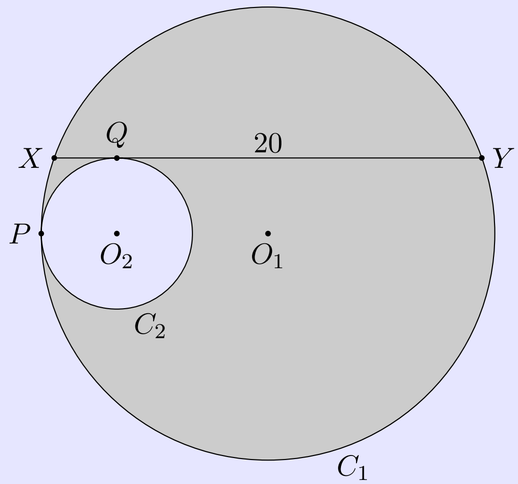
$$\left\lfloor \frac{50}{3} \right\rfloor + \left\lfloor \frac{50}{3^2} \right\rfloor + \left\lfloor \frac{50}{3^3} \right\rfloor = 16 + 5 + 1 = 22,$$

so $3^{22} \mid 50!$ and $n \leq 22$.

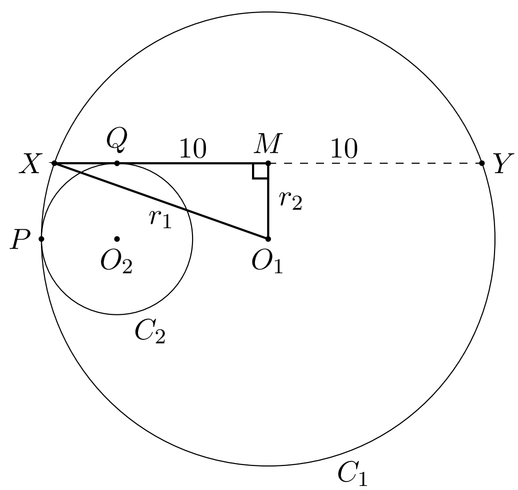
Since $\min(12, 22, 15) = 12$, the largest integer number n is 12.

Answer: $\boxed{12}$.

B3. In the diagram below circles C_1 and C_2 have centres O_1 and O_2. The radii of the circles are respectively r_1 and r_2 with $r_1 = 3r_2$. C_2 is internally tangent to C_1 at P. Chord XY of C_1 has length 20, is tangent to C_2 at Q and is parallel to O_2O_1. Determine the area of the shaded region: that is, the region inside C_1 but not C_2.



Solution: Let M be the midpoint of XY, then $O_1M \perp XY$.



Hence XMO_1 is a right angled triangle and therefore

$$r_1^2 = r_2^2 + 10^2 \tag{1}$$

Clearly the area of the shaded region is $A = \pi(r_1^2 - r_2^2)$, which from (1) gives us

$$A = 100\pi.$$

P.S. Alternatively, one can use $r_1 = 3r_2$ and find the radii explicutly from (1): $9r_2^2 = r_2^2 + 100$, so $r_2 = 10/\sqrt{8}$ and $r_1 = 3r_2 = 30\sqrt{8}$. The difference of areas then is $(900/8 - 100/8)\pi = 100\pi$.

Answer: $\boxed{100\pi}$.

B4. Bob and Jane hold identical decks of twelve cards, three of each colour: red, green, yellow, and blue. Bob and Jane shuffle their decks and then take turns dealing one card at a time onto a pile, with Jane going first. Find the probability that Jane deals *all* her red cards before Bob deals *any* of his red cards.

Give your answer in the form of a fraction in lowest terms.

Solution: We may assume the decks consist of 3 red and 9 black cards. We view each shuffled deck as a binary sequence of length 12, with 1's and 0's indicating red and black cards, respectively. We wish to compute the probability that none of the 1's in Bob's sequence come before the rightmost 1 in Jane's sequence. A *valid* pair of sequences can be modelled by a single sequence of length 12 containing either 5 or 6 ones. In both cases, the first three 1's correspond to Jane's red cards while the last three ones correspond to Bob's. (The combined sequence of length 12 will contain 5 1's in case when Bob's first red cards goes into the pile right after the last Jane's red card.)

There are $\binom{12}{5}+\binom{12}{6}=\binom{13}{6}$ valid pairs of sequences out of $\binom{12}{3}^2$ pairs altogether, giving a desired probability of $\binom{13}{6}/\binom{12}{3}^2=\frac{39}{1100}$.

Alternative solution: Consider sequences of 24 positions numbered by odd numbers (1,3,5,.., 23) for Jane's moves and by even (2,4,6,..., 24) for Bob's. We look at the most right position of Jane's 1 and find the number of possibilities for her other two 1's to the left of it and Bob's 1's (among his 12 possible positions) to the right of it. Then we sum up all these cases to get:

$\binom{2}{2}\binom{10}{3}+\binom{3}{2}\binom{9}{3}+\binom{4}{2}\binom{8}{3}+\binom{5}{2}\binom{7}{3}+\binom{6}{2}\binom{6}{3}+\binom{7}{2}\binom{5}{3}+\binom{8}{2}\binom{4}{3}+\binom{9}{2}\binom{3}{3}=\binom{13}{6}$.

Or by explicit calculation of the coefficients:

$1\times 120+3\times 84+6\times 56+10\times 35+20\times 15+21\times 10+28\times 4+36\times 1=1716$.

The total number of possibilities is $\binom{12}{3}^2=220^2=48400$.

So the probability is

$$\frac{1716}{48400}=\frac{429}{12100}=\frac{39}{1100}.$$

Answer: $\boxed{\frac{39}{1100}}$.

Section C – 10 marks each

C1. The function f is defined on the natural numbers $1, 2, 3, \dots$ by $f(1) = 1$ and

$$f(n) = \begin{cases} f\left(\frac{n}{10}\right) & \text{if } 10 \mid n, \\ f(n-1) + 1 & \text{otherwise.} \end{cases}$$

Note: The notation $b \mid a$ means integer number a is divisible by integer number b.

(a) Calculate $f(2019)$.

(b) Determine the maximum value of $f(n)$ for $n \le 2019$.

(c) A new function g is defined by $g(1) = 1$ and

$$g(n) = \begin{cases} g\left(\frac{n}{3}\right) & \text{if } 3 \mid n, \\ g(n-1) + 1 & \text{otherwise.} \end{cases}$$

Determine the maximum value of $g(n)$ for $n \le 100$.

Solution for (a):

$$\begin{aligned} f(2019) &= f(2018) + 1 = f(2017) + 2 = \dots = f(2010) + 9 = f(201) + 9 \\ &= f(200) + 1 + 9 \\ &= f(20) + 1 + 9 \\ &= f(2) + 1 + 9 \\ &= f(1) + 1 + 1 + 9 \\ &= 1 + 1 + 1 + 9 = 12. \end{aligned}$$

In general, we can see that if $n \equiv m \pmod{10}$ then $f(n) = f(n-m) + m$, where $10 \mid (n-m)$. Therefore $f(n)$ gives the sum of the digits of n. Indeed, $f(2019) = 2 + 0 + 1 + 9 = 12$.

Answer: $\boxed{12}$.

Solution for (b):

We need a number where $n \le 2019$, $n = 1000a + 100b + 10c + d$ and $a + b + c + d$ is maximal. If $a = 2$ the maximum we can hope for is $f(2019) = 12$, but if $a = 1$ we have $f(1999) = 28$ which is the maximum.

Answer: $\boxed{28}$.

Solution for (c):

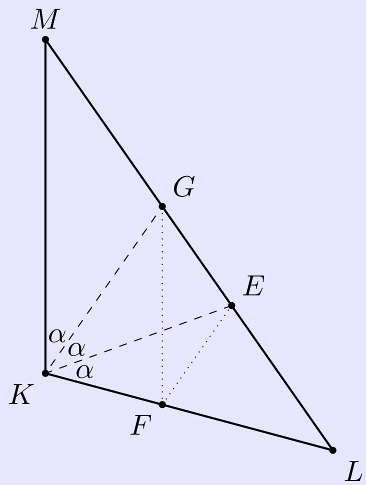
Function g gives the sum of the digits when n is in base 3. Since $100 = 81 + 18 + 1 = 10201_3$, $g(100) = 4$ and g is maximized with $2222_3 = 80$ with $g(80) = 8$.

Answer: $\boxed{8}$.

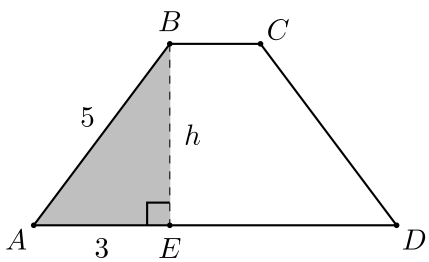
C2. (a) Let $ABCD$ be an isosceles trapezoid with $AB = CD = 5$, $BC = 2$, $AD = 8$. Find the height of the trapezoid and the length of its diagonals.

(b) For the trapezoid introduced in (a), find the exact value of $\cos\angle ABC$.

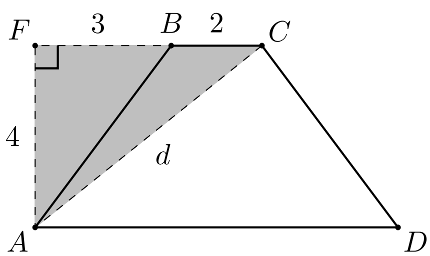
(c) In triangle KLM, let points G and E be on segment LM so that $\angle MKG = \angle GKE = \angle EKL = \alpha$. Let point F be on segment KL so that GF is parallel to KM. Given that $KFEG$ is an isosceles trapezoid and that $\angle KLM = 84°$, determine α.



Solution for (a): Let BE be the height of the trapezoid. Thus, BEA is a right triangle with hypotenuse AB=5 and AE=(8-2)/2=3. Therefore, the height BE=4.



Let AF be the height. From the right triangle AFC, where $AF = 4$, $FC = FB + BC = 3 + 2 = 5$ we find $AC = \sqrt{16 + 25} = \sqrt{41}$.

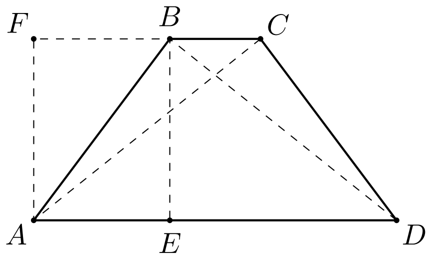


Answer: $\boxed{4 \text{ and } \sqrt{41}}$.

C2. (b) For the trapezoid introduced in (a), find the exact value of $\cos\angle ABC$.

Solution for (b):

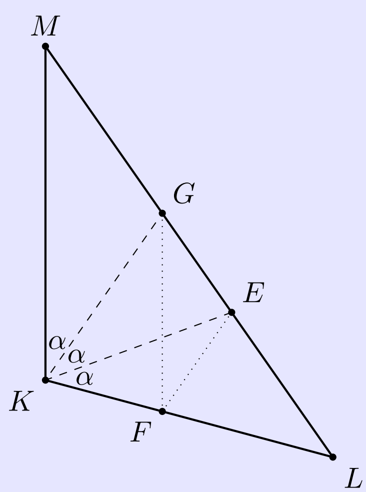
From ABC, using the Law of Cosines we find $41 = 4 + 25 - 20\cos\angle ABC$, so $\cos\angle ABC = -0.6$.



Alternatively, since $FB = AE = 3$, AB=5, $\cos\angle FBA = 3/5$. Then
$\cos\angle ABC = \cos(180 - \angle FBA) = -\cos\angle FBA = -3/5 = -0.6$.

Answer: $\boxed{-0.6}$.

C2. (c) In triangle KLM, let points G and E be on segment LM so that $\angle MKG = \angle GKE = \angle EKL = \alpha$. Let point F be on segment KL so that GF is parallel to KM. Given that $KFEG$ is an isosceles trapezoid and that $\angle KLM = 84°$, determine α.



Solution for (c): Since $KM||FG$ and $KFEG$ is an isosceles trapezoid, we have that $\angle KML = \angle FGE = \angle FKE$, so $\angle LKM = 3\angle KML$. Since the two angles sum up to 96 degrees, we have that $\angle LKM = 72° = 3\alpha$. Thus, $\alpha = 24°$.

Alternatively, since $KFEG$ is an isosceles trapezoid, due to symmetry KLG is an isosceles triangle. Then $2\alpha + 2\alpha + 84° = 180°$, so $\alpha = 24°$.

Answer: $\boxed{24°}$.

C3. Let N be a positive integer. A "good division of N" is a partition of $\{1, 2, \ldots, N\}$ into two disjoint non-empty subsets S_1 and S_2 such that the sum of the numbers in S_1 equals the product of the numbers in S_2. For example, if $N = 5$, then

$$S_1 = \{3, 5\}, \qquad S_2 = \{1, 2, 4\}$$

would be a good division.

(a) Find a good division of $N = 7$.

(b) Find an N which admits two distinct good divisions.

(c) Show that if $N \geq 5$, then a good division exists.

Solution for (a):

Take $S_1 = \{2, 4, 5, 7\}$ and $S_2 = \{1, 3, 6\}$.

Solution for (b):

Consider taking $S_2 = \{1, x, y\}$ for some $1 < x < y \leq N$, and S_1 the complement of S_2 in $\{1, 2, \ldots, N\}$. This is a good division if and only if

$$\frac{N^2 + N}{2} - 1 - x - y = \sum_{s \in S_1} s = \prod_{s \in S_2} s = xy.$$

This rearranges into

$$(x + 1)(y + 1) = \frac{N(N + 1)}{2}. \tag{2}$$

Similarly, if we take $S_2 = \{x', y'\}$, then we have a good division if and only if

$$\frac{N^2 + N}{2} - x' - y' = \sum_{s \in S_1} s = \prod_{s \in S_2} s = x'y'.$$

This rearranges into

$$(x' + 1)(y' + 1) = \frac{N(N + 1)}{2} + 1.$$

With $N = 10$, we have $(x + 1)(y + 1) = 55 = 5 \times 11$ and $(x' + 1)(y' + 1) = 56 = 7 \times 8$. Thus, $(x, y) = (4, 10)$ and $(x', y') = (6, 7)$, and we get two distinct good divisions:

$$S_1 = \{2, 3, 5, 6, 7, 8, 9\}, \qquad S_2 = \{1, 4, 10\};$$

$$S_1 = \{1, 2, 3, 4, 5, 8, 9, 10\}, \qquad S_2 = \{6, 7\}.$$

Another solution for (b) is to take $S_2 = (1, x, y)$ for some $1 < x < y \leq N$, and S_1 the complement of S_2 in $\{1, 2, \ldots, N\}$. Then find a value of N for which $N(N + 1)/2$ has multiple valid factorizations. For example, $N = 20$ gives: $N(N + 1)/2 = 10 \times 21 = 14 \times 15 = (1 + 9)(1 + 20) = (1 + 13)(1 + 14)$. Thus, $N = 20$ admits two distinct good divisions:

$$S_1 = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}, \qquad S_2 = \{1, 9, 20\};$$

$$S_1 = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20\}, \qquad S_2 = \{1, 13, 14\}.$$

C3. (c) Show that if $N \geq 5$, then a good division exists.

Solution for (c):

Let's use equation (2). If $N \geq 6$ is even, then $x = \frac{N-2}{2}$ and $y = N$ satisfy the equation and $1 < x < y \leq N$. If $N \geq 5$ is odd, then $x = \frac{N-1}{2}$ and $y = N - 1$ satisfy the equation and $1 < x < y \leq N$. Thus all $N \geq 5$ admit a good division.

C4. Three players A, B and C sit around a circle to play a game in the order $A \to B \to C \to A \to \cdots$. On their turn, if a player has an even number of coins, they pass half of them to the next player and keep the other half. If they have an odd number, they discard 1 and keep the rest. For example, if players A, B and C start with $(\underline{2}, 3, 1)$ coins, respectively, then they will have $(1, \underline{4}, 1)$ after A moves, $(1, 2, \underline{3})$ after B moves, and $(\underline{1}, 2, 2)$ after C moves, etc. (Here underline indicates the player whose turn is next to move.) We call a position $(\underline{x}, y, z)$ *stable* if it returns to the same position after every 3 moves.

(a) Show that the game starting with $(\underline{1}, 2, 2)$ (A is next to move) eventually reaches $(\underline{0}, 0, 0)$.

(b) Show that any stable position has a total of $4n$ coins for some integer n.

(c) What is the minimum number of coins that is needed to form a position that is neither stable nor eventually leading to $(\underline{0}, 0, 0)$?

Solution for (a):

The game from $(\underline{1}, 2, 2)$ continues to

$$(0, \underline{2}, 2) \to (0, 1, \underline{3}) \to (\underline{0}, 1, 2) \to (0, \underline{1}, 2) \to (0, 0, \underline{2}) \to$$
$$(\underline{1}, 0, 1) \to (0, \underline{0}, 1) \to (0, 0, \underline{1}) \to (\underline{0}, 0, 0).$$

Solution for (b):

Suppose $(\underline{a}, b, c)$ is a stable position (with A moving next). After three moves, we will not have removed a coin, so a must be even. Write $a = 2n$. Then after A moves, we are at $(n, \underline{n+b}, c)$. We must have $n + b$ even, and after removing half of $n + b$, we must leave b remaining in order to be stable. Thus $b = n$. Hence after B moves we have $(n, n, \underline{n+c})$, and applying a similar argument to C gives $c = n$. Thus, the stable positions are some permutation of $(\underline{2n}, n, n)$, where the next player to move has the largest number of coins. Therefore, a stable position has $4n$ coins.

C4. (c) What is the minimum number of coins that is needed to form a position that is neither stable nor eventually leading to $(\underline{0}, 0, 0)$?

Solution for (c): Answer: $\boxed{9}$.

For example, $(\underline{3}, 4, 2)$ is not stable, but after A moves we have $(2, \underline{4}, 2)$, which is stable by (b).

To prove that there is no such a position with fewer number of coins, note the following ideas.

Without loss of generality, we will consider positions $(\underline{a}, b, c)$, where $a \geq 0$, $b \geq 0$, $c \geq 0$.

1. For any initial configuration of coins $(\underline{a}, b, c)$ only finite number of positions produced in the game is possible. Indeed, if no coin is removed during the game, the number of positions is bounded by the number of non-negative partitions of $N = a + b + c$. If coins are being removed, we have a finite number of partitions for each N, $N - 1$, $N - 2$ etc. so the total upper bound is finite.

2. If $(\underline{a}, b, c)$ returns to itself in $K > 0$ moves it is a part of a cycle. If $(\underline{a}, b, c)$ does not lead to a cycle (or is not a part of a cycle) then it will eventually become $(\underline{0}, 0, 0)$. We will identify possible candidates $(\underline{a}, b, c)$ for being a part of a cycle and check them individually. (Checking all positions $(\underline{a}, b, c)$ with $a + b + c \leq 8$ is possible but time consuming!)

Note that $(\underline{0}, 0, 0)$ is stable and we exclude this case from further considerations.

Observation 1. If one or two of the numbers are 0, the triple $(\underline{a}, b, c)$ can not be a part of a cycle because in a cycle 0 is replaced by a positive number, but 0 can not result from halving a positive number.

Observation 2. If a is odd then $(\underline{a}, b, c)$ is not a part of a cycle as after A moves it becomes $(a - 1, \underline{b}, c)$, losing a coin. Similarly, if b is odd then $(a, \underline{b}, c)$ is not a part of a cycle as after B moves it becomes $(a, b - 1, \underline{c})$, losing a coin. Finally, if c is odd then $(a, b, \underline{c})$ is not a part of a cycle as after C moves it becomes $(\underline{a}, b, c - 1)$, losing a coin.

Observation 3. Consider the largest number of coins that appears among any of the players in a cycle. We note that when it occurs, it must be that player's turn, as otherwise they would receive more coins on the previous player's turn, contradicting maximality.

Therefore, the candidates $(\underline{a}, b, c)$ for being a part of a cycle are:

$(\underline{6}, 1, 1)$, $(\underline{4}, 1, 3)$, $(\underline{4}, 2, 2)$, $(\underline{4}, 3, 1)$, $(\underline{4}, 1, 2)$, $(\underline{4}, 2, 1)$, $(\underline{4}, 1, 1)$, $(\underline{2}, 1, 1)$.

We find that

$$(\underline{6}, 1, 1) \to (3, \underline{4}, 1) \to (3, 2, \underline{3}),$$

so it is not a part of a cycle by Observation 2. In $(\underline{4}, x, y)$, if x is odd then after A moves we have $(2, \underline{x + 2}, y)$, where $x + 2$ is odd, so it is not a part of a cycle by Observation 2. This rules all of them out except $(\underline{4}, 2, 2)$, which is stable, and $(\underline{4}, 2, 1)$, which goes

$$(\underline{4}, 2, 1) \to (2, \underline{4}, 1) \to (2, 2, \underline{3}),$$

and hence not a part of a cycle by Observation 2. The remaining case is $(\underline{2}, 1, 1)$, which is stable. Thus all cycles with at most 8 coins are stable.

3. By considering the previous move by the third player to get to $(\underline{4}, 2, 2)$, we see that there are two ways to reach it: from a stable position $(2, 2, \underline{4})$ or from $(4, 2, \underline{3})$.

4. By considering the previous move by the third player to get to $(\underline{2}, 1, 1)$, we see that the only way to reach this stable position is to come from a stable position. Therefore for $N \leq 8$ we could have either stable position or a sequence of positions with non-increasing sum, eventually reaching $(\underline{0}, 0, 0)$.

Note: Although it is not required for the solution, we remark that if any configuration of coins $(\underline{a}, b, c)$ returns to the same position $(\underline{a}, b, c)$ after $3n$ moves for some natural number n, then it in fact returns to the same position after exactly 3 moves. We are not aware of an elementary proof of this fact; our proof is based on computing the eigenvalues of a matrix.

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