Canadian Mathematical Society

2018 Canadian Open Mathematics Challenge

A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

CMS SMC

Expertise. Insight. Solutions.

SOCIETY OF ACTUARIES

Student Identification

Please print clearly and complete all information below. Failure to print legibly or provide complete information may result in your exam being disqualified. This exam is not considered valid unless it is accompanied by your test supervisor's signed declaration.

1. Given/First Name: (Required)

2. Sur-/Last Name: (Required)

3. Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since at least September 15th of this year? (Required for qualification) ☐ **Yes** ☐ No

4. Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of your current address)? (Required for qualification) ☐ **Yes** ☐ No

5. Date of Birth: (Required)

Y	Y	Y	Y	M	M	D	D

6. Grade: (Required)

☐ 8 ☐ 10 ☐ 12
☐ 9 ☐ 11 ☐ Cégep
other: ______________

7. Gender: (Optional)

☐ Female
☐ Male

8. Email Address: (Optional)

DO NOT PHOTOCOPY BLANK EXAMS!
Each page of each copy is uniquely pre-coded to facilitate computer-assisted marking.

Question A1 (4 points)

Suppose x is a real number such that $x(x+3) = 154$. Determine the value of $(x+1)(x+2)$.

Your solution:

Your final answer:

Question A2 (4 points)

Let v, w, x, y, and z be five distinct integers such that $45 = v \times w \times x \times y \times z$. What is the sum of the integers?

Your solution:

Your final answer:

Question A3 (4 points)

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Points $(0, 0)$ and $(3\sqrt{7}, 7\sqrt{3})$ are the endpoints of a diameter of circle Γ. Determine the other x intercept of Γ.

Your final answer:

Question A4 (4 points)

In the sequence of positive integers, starting with $2018, 121, 16, ...$ each term is the square of the sum of digits of the previous term. What is the 2018^{th} term of the sequence?

Your solution:

Your final answer:

Question B1 (6 points)

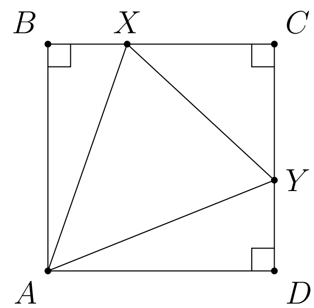
Let $(1+\sqrt{2})^5 = a + b\sqrt{2}$, where a and b are positive integers. Determine the value of $a+b$.

Your solution:

Your final answer:

Question B2 (6 points)

Let $ABCD$ be a square with side length 1. Points X and Y are on sides BC and CD respectively such that the areas of triangles ABX, XCY, and YDA are equal. Find the ratio of the area of ΔAXY to the area of ΔXCY.



Your solution:

Your final answer:

Question B3 (6 points)

The *doubling sum* function is defined by

$$D(a,n) = \overbrace{a + 2a + 4a + 8a + \dots}^{n \text{ terms}}.$$

For example, we have

$$D(5,3) = 5 + 10 + 20 = 35$$

and

$$D(11,5) = 11 + 22 + 44 + 88 + 176 = 341.$$

Determine the smallest positive integer n such that for every integer i between 1 and 6, inclusive, there exists a positive integer a_i such that $D(a_i, i) = n$.

Your solution:

Your final answer:

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Question B4 (6 points)

Determine the number of 5-tuples of integers $(x_1, x_2, x_3, x_4, x_5)$ such that

(a) $x_i \geq i$ for $1 \leq i \leq 5$;

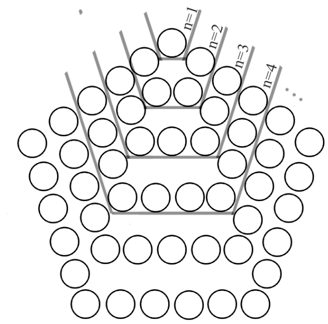
(b) $\sum_{i=1}^{5} x_i = 25$.

Your solution:

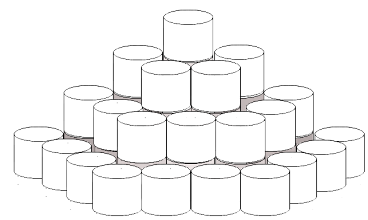
Your final answer:

Question C1 (10 points)

At Math-e^e-Mart, cans of cat food are arranged in an pentagonal pyramid of 15 layers high, with 1 can in the top layer, 5 cans in the second layer, 12 cans in the third layer, 22 cans in the fourth layer etc, so that the k^{th} layer is a pentagon with k cans on each side.



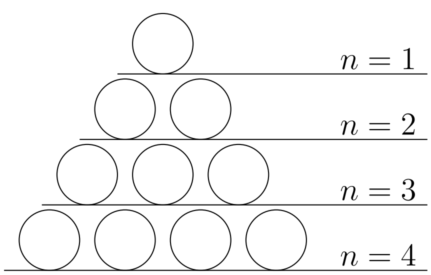
top view


front view

(a) How many cans are on the bottom, 15^{th}, layer of this pyramid?

(b) The pentagonal pyramid is rearranged into a prism consisting of 15 identical layers. How many cans are on the bottom layer of the prism?

(c) A triangular prism consist of identical layers, each of which has a shape of a triangle. (The number of cans in a triangular layer is one of the triangular numbers: 1,3,6,10,...) For example, a prism could be composed of the following layers:



Prove that a pentagonal pyramid of cans with any number of layers $l \geq 2$ can be rearranged (without a deficit or leftover) into a triangular prism of cans with the same number of layers l.

Your solution:

Question C2 (10 points)

Alice has two boxes A and B. Initially box A contains n coins and box B is empty. On each turn, she may either move a coin from box A to box B, or remove k coins from box A, where k is the current number of coins in box B. She wins when box A is empty.

(a) If initially box A contains 6 coins, show that Alice can win in 4 turns.

(b) If initially box A contains 31 coins, show that Alice cannot win in 10 turns.

(c) What is the minimum number of turns needed for Alice to win if box A initially contains 2018 coins?

Your solution:

Question C3 (10 points)

Consider a convex quadrilateral $ABCD$. Let rays BA and CD intersect at E, rays DA and CB intersect at F, and the diagonals AC and BD intersect at G. It is given that the triangles DBF and DBE have the same area.

(a) Prove that EF and BD are parallel.

(b) Prove that G is the midpoint of BD.

(c) Given that the area of triangle ABD is 4 and the area of triangle CBD is 6, compute the area of triangle EFG.

Your solution:

Question C4 (10 points)

Given a positive integer N, Matt writes N in decimal on a blackboard, without writing any of the leading 0s. Every minute he takes two consecutive digits, erases them, and replaces them with the last digit of their product. Any leading zeroes created this way are also erased. He repeats this process for as long as he likes. We call the positive integer M *obtainable* from N if starting from N, there is a finite sequence of moves that Matt can make to produce the number M. For example, 10 is obtainable from 251023 via

$$2510\underline{23} \to \underline{25}106 \to 1\underline{06} \to 10$$

(a) Show that 2018 is obtainable from 2567777899.

(b) Find two positive integers A and B for which there is no positive integer C such that both A and B are obtainable from C.

(c) Let S be any finite set of positive integers, none of which contains the digit 5 in its decimal representation. Prove that there exists a positive integer N for which all elements of S are obtainable from N.

Your solution:

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STUDENT INSTRUCTIONS

General Instructions:

1) Do not open the exam booklet until instructed to do so by your proctor (supervising teacher).

2) **Before the exam time starts, the proctor will give you a few minutes to fill in the Participant Identification on the cover page of the exam.** You don't need to rush. Be sure to fill in all required information fields and write legibly.

3) **Readability counts:** Make sure the pencil(s) you use are dark enough to be clearly legible throughout your exam solutions.

4) Once you have completed the exam and given it to the proctor/teacher you may leave the room.

5) The questions and solutions of the COMC exam must not be publicly discussed or shared (including online) for at least 24 hours.

Mobile phones and calculators are NOT permitted.

Exam Format:

There are three parts to the COMC to be completed in a total of 2 hours and 30 minutes:

PART A: Four introductory questions worth 4 marks each. You do not have to show your work. A correct final answer gives full marks. However, if your final answer is incorrect and you have shown your work in the space provided, you might earn partial marks.

PART B: Four more challenging questions worth 6 marks each. Marking and partial marks follow the same rule as part A.

PART C: Four long-form proof problems worth 10 marks each. Complete work must be shown. Partial marks may be awarded.

Diagrams provided are *not* drawn to scale; they are intended as aids only.

Scrap paper/extra pages: You *may* use scrap paper, but you have to throw it away when you finish your work and hand in your booklet. Only the work you do on the pages provided in the booklet will be evaluated for marking. Extra pages are not permitted to be inserted in your booklet.

Exact solutions: It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2 + \sqrt{7}$, etc., rather than as 12.566, 4.646, etc.

Awards: The names of all award winners will be published on the Canadian Mathematical Society website.