Canadian Open Mathematics Challenge 2017

Official Solutions

Presented by the Canadian Mathematical Society and supported by the Actuarial Profession.

The COMC has three sections:

A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

Some novel solutions were selected from students' papers in order to showcase other possible ways of reasoning.

- David Rowe of Holy Heart of Mary Regional High School
- Haneul Shin of Bergen County Academies
- Victor Wang of Sir Winston Churchill Secondary School
- Freddie Zhao of Indus Center for Academic Excellence

COMC exams from other years, with or without the solutions included, are free to download online. Please visit http://comc.math.ca/2017/practice.html

Section A – 4 marks each

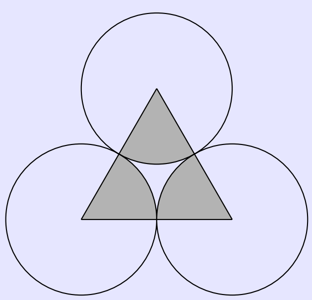
A1. The average of the numbers $2, 5, x, 14, 15$ is x. Determine the value of x.

Correct answer: $\boxed{9}$.

Solution 1: The average of the numbers is $\dfrac{2+5+x+14+15}{5} = \dfrac{x+36}{5}$; this must be equal x. Solving $36 + x = 5x$, we have $x = 9$.

Solution 2: Since the x being there would not affect the average of the other 4 numbers, $x = \dfrac{2+5+14+15}{4} = \dfrac{36}{4} = 9$.

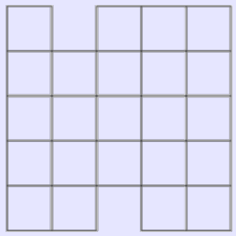
A2. An equilateral triangle has sides of length 4cm. At each vertex, a circle with radius 2cm is drawn, as shown in the figure below. The total area of the shaded regions of the three circles is $a \times \pi$ cm^2. Determine a.



Correct answer: $\boxed{2}$.

Solution: The area of the intersection of each circle and the triangle is $4\pi/6$ cm^2. The three circles do not overlap, thus the total area is 2π cm^2.

A3. Two 1×1 squares are removed from a 5×5 grid as shown.



Determine the total number of squares of various sizes on the grid.

Correct answer: $\boxed{\mathbf{39}}$ **squares.**

Solution 1: There are 23 1×1 squares, 12 2×2 squares, and 4 3×3 squares. This gives $23 + 12 + 4 = 39$ squares.

Solution 2: Number of squares without missing parts is $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$.

Missing 1×1 squares is 2.

Missing 2×2 squares is 4.

Missing 3×3 squares is 5.

Missing 4×4 squares is 4.

Missing 5×5 squares is 1.

Total missing squares is $2+4+5+4+1 = 16$. Thus the number of squares in the diagram is $55 - 16 = 39$.

A4. Three positive integers a, b, c satisfy

$$4^a \times 5^b \times 6^c = 8^8 \times 9^9 \times 10^{10}.$$

Determine the sum of $a + b + c$.

Correct answer: $\boxed{\mathbf{36}}$**.**

Solution: The prime factorization of the left-hand side is

$$2^{2a} \times 5^b \times 2^c \times 3^c = 2^{2a+c} \times 3^c \times 5^b,$$

and the prime factorization of the right-hand-side is

$$2^{24} \times 3^{18} \times 2^{10} \times 5^{10} = 2^{34} \times 3^{18} \times 5^{10}.$$

Since the prime factorization of an integer is determined uniquely, we have, $2a + c = 34$, $c = 18$ and $b = 10$. We now find a. Since $2a + c = 34$ and $c = 18$, $2a + 18 = 34$, yielding $a = 8$. Therefore, $a + b + c = 8 + 10 + 18 = 36$.

Section B – 6 marks each

B1. Andrew and Beatrice practice their free throws in basketball. One day, they attempted a total of 105 free throws between them, with each person taking at least one free throw. If Andrew made exactly 1/3 of his free throw attempts and Beatrice made exactly 3/5 of her free throw attempts, what is the highest number of successful free throws they could have made between them?

Correct answer: **59** **free throws.**

Solution 1: From their success rates we conclude that each of them must have made a multiple of 15 throws. Specifically, from Andrew's success rate, his number of throws must be a multiple of 3. Since the total number of throws (105) is also a multiple of 3, Beatrice's number of throws must be a multiple of 3 too. From Beatrice's success rate, her number of throws must be a multiple of 5, and thus must in fact be a multiple of 15. Similarly, since 105 is a multiple of 5, Andrew's number of throws must be a multiple of 5 and thus a multiple of 15 too.

Since $1/3 < 3/5$, to maximize the result we should assume that Andrew made the least possible number of throws, that is 15. Then Beatrice made 90 throws.

Then the number of successful free throws they could have made between them is
$15 \times \frac{1}{3} + 90 \times \frac{3}{5} = 5 + 54 = 59$.
The maximum possible number of successful free throws they could have made between them is 59.

Solution 2: Suppose Andrew made a free throws and Beatrice b free throws, then $a + b = 105$, $a > 0$, $b > 0$. Let M be the number of successful free throws. We have

$$M = \frac{a}{3} + \frac{3b}{5} = \frac{a}{3} + \frac{3(105 - a)}{5} = \frac{945 - 4a}{15} = 63 - \frac{4a}{15}.$$

M is maximal when $\frac{4a}{15}$ is minimal. That is, $a = 15$ and so $M = 59$.

The maximum possible number of successful free throws they could have made between them is 59.

B2. There are twenty people in a room, with a men and b women. Each pair of men shakes hands, and each pair of women shakes hands, but there are no handshakes between a man and a woman. The total number of handshakes is 106. Determine the value of $a \times b$.

Correct answer: $\boxed{\mathbf{84}}$**.**

Solution 1: Since there are twenty people in the room,

$$a + b = 20.$$

Now we use the fact that there are no handshakes between a man and a woman. In a group of m people, there are $m(m-1)/2$ pairs of people. Hence, the number of handshakes that took place is

$$\frac{a(a-1)}{2} + \frac{b(b-1)}{2} = 106,$$

which simplifies to

$$\frac{a^2 + b^2 - (a+b)}{2} = 106 \Rightarrow a^2 + b^2 = 212 + (a+b) = 232.$$

Substituting $b = 20 - a$ into this equation yields

$$a^2 + (20-a)^2 = 232 \Rightarrow a^2 + (400 - 40a + a^2) = 232$$

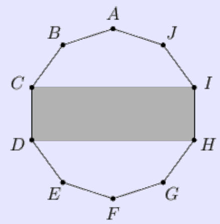
$$\Rightarrow 2a^2 - 40a + 168 = 0 \Rightarrow 2(a^2 - 20a + 84) = 0.$$

This factors as $2(a-14)(a-6) = 0$. Therefore, $a = 14$ or $a = 6$. Since $a + b = 20$, $(a, b) = (14, 6)$ or $(6, 14)$. Hence, $a \times b = 84$.

Solution 2: Since there are 20 people in the room, there are $\frac{20 \times 19}{2} = 190$ pairs of people. Out of them exactly $a \times b$ pairs do not shake hands.

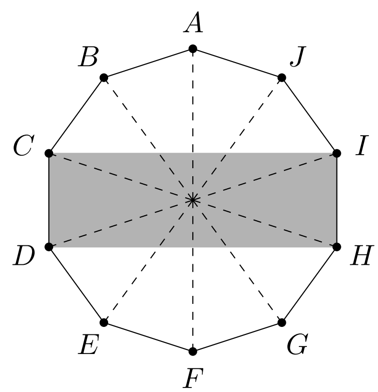
Thus, we have $190 - a \times b = 106$, and so $a \times b = 190 - 106 = 84$.

B3. Regular decagon (10-sided polygon) $ABCDEFGHIJ$ has area 2017 square units. Determine the area (in square units) of the rectangle $CDHI$.



Correct answer: **806.8** **square units.**

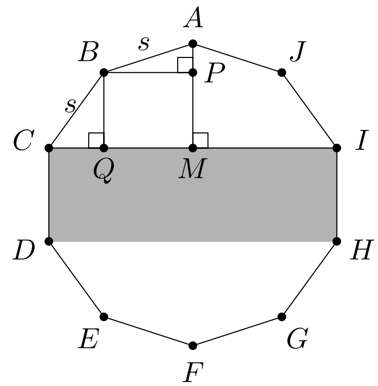
Solution 1: If O is the centre of the decagon and we cut the decagon into equal isosceles triangles, as shown, we can see that the area of the decagon is 10 times the area of one triangle. Since diagonals CH and DI are equal in length and bisect each other, quadrilateral $CDHI$ is a rectangle, with the same base as each of the triangles, but twice the height. Thus the area of $CDHI$ is 4 times the area of each of the triangles and hence 40% of the area of the decagon or $0.4 \times 2017 = 806.8$ square units.



The area of the rectangle is 806.8 square units.

A second solution is provided on the next page.

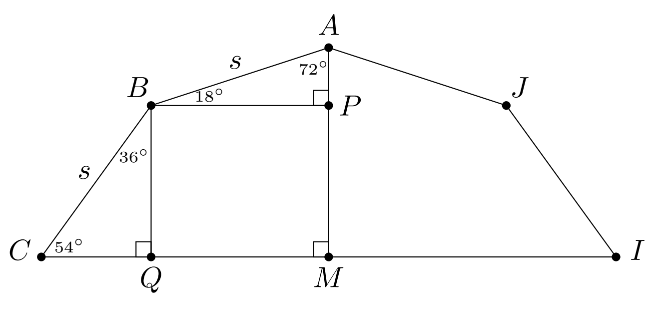
Solution 2: Let $s = |AB|$ be the side length of the regular decagon. Since it is a regular polygon with 10 sides, all its angles are equal, in particular $\angle ABC = \angle JAB = 180 \times (10-2)/10 = 144°$. Let M be the midpoint of CI. Draw AM, which will be perpendicular to CI. Place point P on AM and point Q on CM such that BP is perpendicular to AM and BQ is perpendicular to CM.



The area of $ABCM$ could be found as a sum of areas of two right triangles BPA and BQC and the rectangle $BPMQ$. Note that angle BAP is 144/2=72° and so angle ABP is $90 - 72 = 18°$. Now, angle CBQ is $144 - 18 - 90 = 36°$, and so angle BCQ is $90 - 36 = 54°$.

The area of the right triangle BPA is $\frac{1}{2}s^2 \sin 18° \cos 18° = \frac{1}{4}s^2 \sin 36°$.

The area of the right triangle BQC is $\frac{1}{2}s^2 \sin 54° \cos 54° = \frac{1}{4}s^2 \sin 108° = \frac{1}{4}s^2 \cos 18°$.



The area of the rectangle $BPMQ$ is $s^2 \sin 54° \cos 18° = \frac{1}{2}s^2(\sin 36° + \sin 72°) = \frac{1}{2}s^2(\sin 36° + \cos 18°)$.

Therefore, the area of $ABCM$ is $\frac{3}{4}s^2(\sin 36° + \cos 18°) \equiv \Delta$.

Next, the area of $CDHI$ is $2s^2(\sin 36° + \cos 18°) = \frac{8}{3}\Delta$.

The total area of the decagon is $4\Delta + \frac{8}{3}\Delta = \frac{20}{3}\Delta = 2017$.

Thus, the area of $CDHI$ is $\frac{8}{20} \times 2017 = 806.8$.

B4. Numbers a, b and c form an arithmetic sequence if $b - a = c - b$. Let a, b, c be positive integers forming an arithmetic sequence with $a < b < c$. Let $f(x) = ax^2 + bx + c$. Two distinct real numbers r and s satisfy $f(r) = s$ and $f(s) = r$. If $rs = 2017$, determine the smallest possible value of a.

Correct answer: $\boxed{9}$.

Solution 1: Note that

$$\begin{aligned} ar^2 + br + c &= s \qquad (1) \\ as^2 + bs + c &= r \qquad (2) \end{aligned}$$

Subtracting the second equation from the first yields

$$a(r^2 - s^2) + b(r - s) = -(r - s) \Rightarrow a(r + s)(r - s) + (b + 1)(r - s) = 0$$

$$\Rightarrow (a(r + s) + b + 1)(r - s) = 0.$$

Since $r \neq s$, $r + s = -\frac{b+1}{a}$.
Substituting $s = -\frac{b+1}{a} - r$ into the first equation yields
$ar^2 + br + c = -\frac{b+1}{a} - r \Rightarrow ar^2 + (b + 1)r + c + \frac{b+1}{a} = 0$. Substituting $r = -\frac{b+1}{a} - s$ into the second equation yields $as^2 + bs + c = -\frac{b+1}{a} - s \Rightarrow as^2 + (b + 1)s + c + \frac{b+1}{a} = 0$. Therefore, r, s are the roots to the equation $ax^2 + (b + 1)x + c + \frac{b+1}{a} = 0$. The product of the roots of this equation is $\frac{c}{a} + \frac{b+1}{a^2} = 2017$.

We also know that the coefficients form an arithmetic sequence, so let $b = a + k, c = a + 2k$. Then we have $\frac{a+2k}{a} + \frac{a+k+1}{a^2} = 2017$ and so

$$k = \frac{2016a^2 - 1 - a}{2a + 1} = 1008a - 504 + \frac{503 - a}{2a + 1} = 1008a - 504 + \frac{1}{2}\left(\frac{1007}{2a + 1} - 1\right).$$

Thus, in order for k to be integer, $2a + 1$ must be a factor of $1007 = 19 \times 53$. Thus, $a = 9$, $a = 26$ or $a = 503$.

The smallest positive integer a for which k is an integer is $a = 9$.

For completeness, we find $k = 8594$ and so $b = 8603$ and $c = 17197$.

Then, $r = -478 + 3\sqrt{25163}$ and $s = -478 - 3\sqrt{25163}$ satisfy the above relations.

The answer is 9.

A second solution is provided on the next page.

Solution 2: Note that

$$\begin{aligned} ar^2 + br + c &= s \qquad (1) \\ as^2 + bs + c &= r \qquad (2) \end{aligned}$$

Subtracting the second equation from the first yields

$$a(r^2 - s^2) + b(r - s) = -(r - s) \Rightarrow a(r + s)(r - s) + (b + 1)(r - s) = 0$$

$$\Rightarrow (a(r + s) + b + 1)(r - s) = 0,$$

since $r \neq s$, $r + s = -\frac{b+1}{a}$.

Adding (1) and (2) we get

$$a((r + s)^2 - 2rs) + b(r + s) + 2c = r + s.$$

Using $r + s = -\frac{b+1}{a}$ and $rs = 2017$, we obtain

$$\frac{(b + 1)^2}{a} - 2a \cdot 2017 - \frac{(b + 1)(b - 1)}{a} + 2c = 0$$

and then

$$\frac{2(b + 1)}{a} - 2a \cdot 2017 + 2c = 0,$$

or equivalently

$$b + 1 - 2017a^2 + ac = 0.$$

Substituting $c = 2b - a$, we get

$$b + 1 - 2017a^2 + 2ab - a^2 = 0,$$

and so

$$b = \frac{2018a^2 - 1}{2a + 1} = \frac{(2a + 1)(1009a - 1) - 1007a}{2a + 1} = 1009a - 1 - \frac{1007a}{2a + 1}.$$

Now, b is integer, so $\frac{1007a}{2a+1}$ must be integer. Since $\gcd(a, 2a + 1) = 1$, $2a + 1$ must be a factor of $1007 = 19 \times 53$, and so $a = 9$, $a = 26$ or $a = 503$. The smallest value is $a = 9$.

Section C – 10 marks each

Note: Section C questions require participants to show all their work.

C1. For a positive integer n, we define function $P(n)$ to be the sum of the digits of n **plus** the number of digits of n. For example, $P(45) = 4+5+2 = 11$. (Note that the first digit of n reading from left to right, cannot be 0).

(a) Determine $P(2017)$.

Solution: $2+0+1+7+4 = 14$.

The answer is $\boxed{\text{P(2017)=14}}$.

(b) Determine all numbers n such that $P(n) = 4$.

Solution: Consider 1-digit numbers, then $n = 3$. Among 2-digit numbers we need those with the sum of their digits equal to 2; so we have $n = 11$ and $n = 20$. Among 3-digit numbers we need those with the sum of their digits equal to 1; so we have $n = 100$. For numbers with 4 or greater than 4 digits, $P(n) > 4$, so we have listed all possible numbers with the required property.

The answer is $\boxed{3, 11, 20, 100}$.

(c) Determine with an explanation whether there exists a number n for which $P(n) - P(n+1) > 50$.

Solution: If n and $n+1$ differ only in one last digit the equality $P(n) - P(n+1) > 50$ is not possible.

Consider the case when n consisits of k 9's. Then $P(n) = 9k + k = 10k$ and $P(n+1) = 1 + (k+1) = k+2$. So we have $P(n) - P(n+1) = 9k - 2 > 50$, so $k \geq 6$. For $k = 6$ we obtain $n = 999,999$ and $P(999,999) - P(1,000,000) = 60 - 8 > 50$.

The answer is $\boxed{\text{Yes, for example}\, n = 999,999}$.

C2. A function $f(x)$ is periodic with period $T > 0$ if $f(x+T) = f(x)$ for all x. The smallest such number T is called the least period. For example, the functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π.

(a) Let a function $g(x)$ be periodic with the least period $T = \pi$. Determine the least period of $g(x/3)$.

Solution: A period of $g\left(\frac{x}{3}\right)$ is 3π because $g\left(\frac{x+3\pi}{3}\right) = g\left(\frac{x}{3}+\pi\right) = g\left(\frac{x}{3}\right)$. Note that if $g(x/3)$ had a period smaller than 3π, then $g(x)$ would have a period smaller than π, a contradiction.

The answer is $\boxed{3\pi}$.

(b) Determine the least period of $H(x) = \sin(8x) + \cos(4x)$.

Solution: The least period of $\sin(8x)$ is $\frac{\pi}{4}$ because $\sin(8(x+\frac{\pi}{4})) = \sin(8x+2\pi) = \sin(8x)$. The least period of $\cos(4x)$ is $\frac{\pi}{2}$ because $\cos(4(x+\frac{\pi}{2})) = \cos(4x+2\pi) = \cos(4x)$. We pick the larger of the two periods for the sum of the functions. In general, it is the least common multiple of the two periods that must be picked.

Note that if $H(x)$ had a period smaller than $\pi/2$, then $\cos(4x) = H(x) - \sin(8x)$ would also have a period smaller than $\pi/2$, and so $\cos x$ would have a period smaller than 2π. Indeed, if $\cos(4x) = \cos(4(x+T)) = \cos(4x+4T)$ and $T < \pi/2$ then $\cos y = \cos(y+4T)$ and $4T < 4(\pi/2) = 2\pi$. This is a contradiction.

The answer is $\boxed{\pi/2}$.

(c) Determine the least periods of each of $G(x) = \sin(\cos(x))$ and $F(x) = \cos(\sin(x))$.

Solution in two parts:

1. $G(x+2\pi) = \sin(\cos(x+2\pi)) = \sin(\cos(x)) = G(x)$.

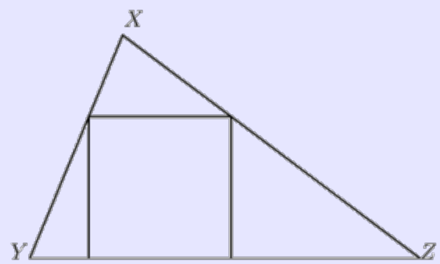
To prove that $T = 2\pi$ is the smallest positive number consider for example $x = 0$. Then $G(0) = \sin(1)$. For $G(T) = \sin(\cos(T)) = \sin(1)$, we need either $\cos(T) = 1 + 2\pi k$ or $\cos(T) = -1 + (2k+1)\pi$, $k \in \mathbf{Z}$. In the first case we see that the inequality $-1 \le 1 + 2\pi k \le 1$ is true only for $k = 0$. This gives $\cos(T) = 1$, and so the smallest positive $T = 2\pi$. In the second case the inequality $-1 \le -1 + (2k+1)\pi \le 1$ is true for no integer k. Thus, the only solution is $T = 2\pi$.

2. $F(x+\pi) = \cos(\sin(x+\pi)) = \cos(-\sin(x)) = \cos(\sin(x)) = F(x)$.

To prove that $T = \pi$ is the smallest positive number consider for example $x = 0$. Then $F(0) = 1$. Now, we need $F(0+T) = \cos(\sin(T)) = 1$. Thus, $\sin T = 0$, and so the smallest positive $T = \pi$.

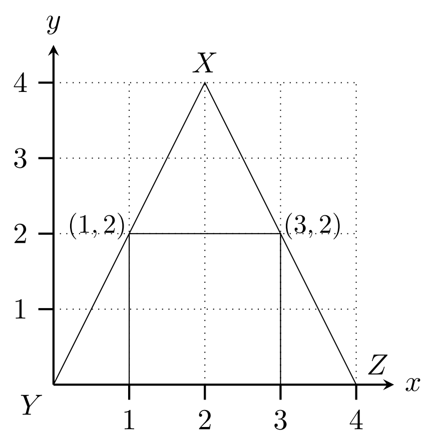
The answers are $\boxed{2\pi}$ and $\boxed{\pi}$.

C3. Let XYZ be an acute-angled triangle. Let s be the side-length of the square which has two adjacent vertices on side YZ, one vertex on side XY and one vertex on side XZ. Let h be the distance from X to the side YZ and let b be the distance from Y to Z.



(a) If the vertices have coordinates $X = (2, 4)$, $Y = (0, 0)$ and $Z = (4, 0)$, find b, h and s.

Solution: Subtracting coordinates of corresponding points we have $b = z_1 - y_1 = 4 - 0 = 4$, $h = x_2 - y_2 = x_2 - z_2 = 4 - 0 = 4$. Note that points $(1, 2)$ and $(3, 2)$ lie on sides XY and XZ respectively and together with points $(1, 0)$ and $(3, 0)$ they define a square that satisfies the conditions of the problem. This square has side $s = 2$. (Alternatively, from similar triangles we have $\frac{s}{b} = \frac{h-s}{h}$ that is $\frac{s}{4} = \frac{4-s}{4}$, so $s = 2$.)



The answer is $\boxed{b = 4, \quad h = 4, \quad s = 2}$.

(b) Given the height $h = 3$ and $s = 2$, find the base b.

Solution: Since PQ is parallel to YZ, triangle XPQ is similar to XYZ. Because $h = 3$ and corresponding height of XPQ is $3 - 2 = 1$, we conclude that the base of XYZ is $3 \times 2 = 6$.

The answer is $\boxed{b = 6}$.

(c) If the area of the square is 2017, determine the minimum area of triangle XYZ.

Solution 1: From the similarity of XPQ and XYZ we have $\frac{s}{b} = \frac{h-s}{h}$. Equivalently, $s = \frac{bh}{b+h}$.

Thus, $s^2 = \frac{(bh)^2}{(b+h)^2} = 2K\frac{bh}{(b+h)^2}$. Here $K = \frac{bh}{2}$ is the area of XYZ.

By AM-GM inequality (arithmetic mean is greater or equal than geometric mean), we have $\frac{4bh}{(b+h)^2} \leq 1$, thus $s^2 \leq \frac{K}{2}$.

We have $2017 \leq \frac{K}{2}$. Thus, $4034 \leq K$.

Now, we show that the minimum area is achieved for $b = h = 2s = 2\sqrt{2017}$. Indeed, if $b = h$ then $s = \frac{bh}{b+h} = \frac{b}{2} = \frac{h}{2}$ and $K = 2s^2 = 4034$.

The minimum value for the area of XYZ is 4034.

The answer is $\boxed{4034}$.

Solution 2: From the similarity of XPQ and XYZ we have $\frac{s}{b} = \frac{h-s}{h}$. Thus $b = \frac{sh}{h-s}$. The area

$$[XYZ] = \frac{bh}{2} = \frac{h^2 s}{2(h-s)}.$$

Finding the minimum value of this expression is equivalent to finding the maximum of its reciprocal $\frac{2(h-s)}{h^2 s} \to \max$.

Now, note that the reciprocal is a quadratic function in the variable $\frac{1}{h}$, that is

$$\frac{2(h-s)}{h^2 s} = -2\left(\frac{1}{h}\right)^2 + \frac{2}{s}\left(\frac{1}{h}\right),$$

so the maximum is achieved at $\frac{1}{h} = \frac{1}{2} \cdot \frac{1}{s}$ or equivalently, for $h = 2s$.

Then $b = 2s$ and the area $[XYZ] = 2s^2 = \boxed{4034}$.

Special thanks to ***Haneul Shin of Bergen County Academies*** *for providing the solution above. Edited for clarity and concision.*

C4. Let n be a positive integer and $S_n = \{1, 2, \ldots, 2n-1, 2n\}$. A *perfect pairing* of S_n is defined to be a partitioning of the $2n$ numbers into n pairs, such that the sum of the two numbers in each pair is a perfect square. For example, if $n = 4$, then a perfect pairing of S_4 is $(1, 8), (2, 7), (3, 6), (4, 5)$. It is not necessary for each pair to sum to the same perfect square.

(a) Show that S_8 has at least one perfect pairing.

Solution: For $n = 8$ we have pairs $(1, 8), (2, 7), (3, 6), (4, 5)$ each of which sums to 9 and $(9, 16), (10, 15), (11, 14), (12, 13)$ each of which sums to 25.

The answer is $\boxed{(1, 8), (2, 7), (3, 6), (4, 5), (9, 16), (10, 15), (11, 14), (12, 13)}$.

(b) Show that S_5 does not have any perfect pairings.

Solution 1: Each pair must have a sum 16, 9, 4. There are 5 pairs that must sum to 55. There must be at least two pairs that sum to 16. If there are two such pairs, then the other three pairs must sum to 55 - 32 = 23, which cannot be done with the numbers 9, 4. There cannot be more than two pairs that sum to 16, since there are only two such pairs, namely (6, 10), (7, 9).

P.S. It is also possible to construct an argument by considering all sums modulo 8:

$$55 \equiv 7 \mod 8, \quad 16 \equiv 0 \mod 8, \quad 9 \equiv 1, \quad \mod 8, \quad 4 \equiv 4 \mod 8.$$

Thus, for five pairs we must have $7 = 4 + 1 + 1 + 1 + 0$. But this does not give the correct sum: $4 + 9 + 9 + 9 + 16 = 47 \neq 55$, so perfect pairing is not possible.

Solution 2: Consider $S_5 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Note that 10 needs 6 to make a perfect pair, which means that 3 can only pair perfectly with 1. This leaves 8 with no partner with which to make a perfect pair.

Special thanks to ***David Rowe of Holy Heart of Mary Reg. H.S.*** *for providing the solution above. Edited for clarity and concision.*

(c) Prove or disprove: there exists a positive integer n for which S_n has at least 2017 different perfect pairings. (Two pairings that are comprised of the same pairs written in a different order are considered the same pairing.)

Solution 1: The easiest way to get a perfect pairing from $\{a, a+1, ..., 2m\}$ is for all pairs to have the same sum, i.e. $a + 2m$ to be a square. The next easiest way is if there are two possible sums, which happens when the sets $\{a, ..., 2n\}$ and$\{2n+1, ..., 2m\}$ have both $a + 2n$ and $2n + 1 + 2m$ being squares.

We will first prove the following lemma:

Lemma: Let $a \equiv 1 \pmod 4$ be a positive integer. Then there exists an even positive integer m such that $2m > a$ and $\{a, a+1, \ldots, 2m\}$ has at least two perfect pairings into $(2m - a + 1)/2$ pairs.

Proof of lemma: Let x, y be odd positive integers satisfying

$$x^2 > 2a, \quad y^2 > 2(x^2 - a + 1).$$

Then taking $m = \frac{y^2 - x^2 + a - 1}{2}$, we have

$$\{a, \ldots, 2m\} = \{a, a+1, \ldots, x^2 - a\} \dot{\cup} \{x^2 - a + 1, x^2 - a + 2, \ldots, y^2 - x^2 + a - 1\},$$

where we pair up elements in the first set to all have sum x^2, and in the second set to have sum y^2. We would also like to pair up elements so every pair has the same sum, i.e. we would like $2m + a$ to be a square. This translates to solving

$$y^2 - x^2 + 2a - 1 = z^2,$$

subject to x, y odd positive integers, z a positive integer, $x^2 > 2a$, and $y^2 > 2(x^2 - a + 1)$. The condition of m being even will be automatically satisfied since $y^2 - x^2 + a - 1 \equiv 0 \pmod 4$ using x, y odd (and so $y^2 - x^2 = (y - x)(y + x) \equiv 0 \pmod 4$) and $a \equiv 1 \pmod 4$. We rearrange our equation into

$$(y - z)(y + z) = y^2 - z^2 = x^2 - 2a + 1.$$

Picking $x = 2r + 1$ to be any odd positive integer bigger than $\sqrt{2a}$, and noting that $x^2 = 4r(r + 1) + 1 \equiv 1 \pmod 8$ and $2a \equiv 2 \pmod 8$, we see that $x^2 - 2a + 1 \equiv 0 \pmod 8$. Thus we set $y - z = 2$, $y + z = \frac{x^2 - 2a + 1}{2}$, and we get

$$y = \frac{x^2 - 2a + 5}{4}, \quad z = \frac{x^2 - 2a - 3}{4}.$$

Hence y, z are positive integers, y is odd, and y, z satisfy $y^2 - x^2 + 2a - 1 = z^2$. We assumed that $x^2 > 2a$, so we will be done as long as $y^2 > 2(x^2 - a + 1)$. But y^2 is a quartic in x, so this inequality is satisfied for all x sufficiently large, and thus we can pick x to be sufficiently large for this to hold. This completes the proof of the lemma.

We now show that for any $N \geq 2$ there exists a positive integer n for which S_n has at least N different perfect pairings. We do it by induction on N, where we also impose that n must be even. The case $N = 2$ is given by the above lemma with $a = 1$. Assume we have it up to $N - 1 \geq 2$, and m is an even number such that $\{1, \ldots, 2m\}$ has at least $N - 1$ perfect pairings. Then take $a = 2m + 1 \equiv 1 \pmod 4$ in the above lemma, to get an $n > m$ even with $\{2m + 1, \ldots, 2n\}$ having at least two perfect pairings. Combining these with partitions of $\{1, \ldots, 2m\}$, we see that $\{1, \ldots, 2n\}$ has at least $2(N - 1) \geq N$ perfect pairings. Therefore the result is true for $N = 2017$.

Solution 2: Let us first prove the following statement.

> **Lemma:** There exist arbitrarily large $n \equiv 0 \mod (4)$ for which the set S_n has a perfect pairing.
>
> **Proof of lemma:** This proof is by induction. We know that a perfect pairing exists for S_4. Suppose we have a perfect pairing of the set S_n for some n divisible by 4. We will construct a perfect pairing of the set S_m with some $m > n$ also divisible by 4.
> The construction is a straightforward generalization of the method used in the numerical example of part (a). We will find $m = n+r$ with $r > 0$ divisible by 4 such that $2n+2m+1 = k^2$ for some integer k. This yields a perfect pairing for S_m consisting of all pairs in a perfect pairing for S_n and new pairs
>
> $$(2n + 1, 2m),\ (2n + 2, 2m - 1),\ \ldots,\ (2n + r, 2n + r + 1).$$
>
> So we have the equation $2(n + m) + 1 = 4n + 2r + 1 = k^2$, i.e.
>
> $$r = \frac{k^2 - 4n - 1}{2}.$$
>
> Put $k = 2q + 1$, where $q^2 \geq n$. Then
>
> $$r = (4q^2 + 4q + 1 - 4n - 1)/2 = 2(q(q + 1) - n). \qquad (*)$$
>
> Since both $q(q + 1)$ and n are even, r is divisible by 4 and the construction is complete. This completes the proof of the lemma.
>
> Note that in this construction each perfect pairing for S_n gives rise to a perfect pairing for S_m and they all are different.

Suppose now, by way of contradiction, that there is some number H (for the purpose of the problem, $H < 2017$) such that for any $i \geq 16$ divisible by 4 the set S_i has at most H perfect pairings. Let $n \geq 16$ be divisible by 4 and such that S_n has exactly H perfect pairing. Carrying out the above construction, we will find H perfect pairings for S_m. We demonstrate below how to obtain S_m that has a perfect pairing different from those provided by our construction. Such an S_m will have at least $H + 1$ perfect pairings, a contradiction.

Since n is divisible by 4, Put $q = n/4$ in $(*)$, which yields $r = n^2/8 + n/2 - 2n = n(n - 12)/8$. Here the role of the condition $n \geq 16$ becomes clear: it ensures that $r > 0$. The following simple calculation shows that $2m+1$ is an integer square:

$$2m+1 = 2(n+r)+1 = 2n + \frac{n(n-12)}{4} + 1 = \frac{n^2-4n+4}{4} = \left(\frac{n-2}{2}\right)^2.$$

Hence, in addition to the perfect pairings for S_m that correspond to those for S_n, we have the perfect pairing

$$\{(1,2m),(2,2m-1),\ldots,(m,m+1)\}.$$

Solution 3: This solution is based on the fact that there exists an arithmetic progression of length 3 consisting of integer squares; specifically, $\{1, 25, 49\}$.

Observe that if $n = \frac{25N^2-1}{2}$ for some odd integer $N \geq 1$ then there exists at least one perfect pairing of the set S_n, namely $(1, 25N^2-1), (2, 25N^2-2), \ldots, (\frac{25N^2-1}{2}, \frac{25N^2+1}{2})$.

Now, we can choose large enough integer N for which there are at least 2016 pairs of integers (a, b) such that $1 \leq a < b \leq 2n$ and $b - a = 24N^2$. (Any odd N with $N^2 > 2017$ will do.)

Then one can take any 2 pairs $(a, 25N^2-a)$ and $(b, 25N^2-b)$ from the perfect pairing above and swap them with $(a, 25N^2-b)$ and $(b, 25N^2-a)$. Whenever $b - a = 24N^2$, this swap will produce a new perfect pairing because $25N^2 + b - a = 25N^2 + 24N^2 = 49N^2$ and $25N^2 - b + a = 25N^2 - 24N^2 = N^2$. Thus we can construct 2016 new perfect pairings, which together with the initial perfect pairing gives 2017 different perfect pairings.

Special thanks to ***Victor Wang of Sir Winston Churchill S.S.*** *for providing the solution above. Edited for clarity and concision.*

Solution 4: Let us split the set S_n into two groups, $1 \ldots x$ and $x+1 \ldots 2n$. The partition $(1, x), (2, x-1), \ldots, (x+1, 2n), (x+2, 2n-1), \ldots)$ of S_n is a perfect pairing provided x is even and $1 + x = m^2$, $m^2 + 2n = k^2$ for some odd integers $k > m > 1$. Conversely, any pair of integers k, m such that $k^2 - m^2 = 2n$ yields a perfect pairing of S_n and the perfect pairings corresponding to different pairs (m, k) are different. We will exhibit an n such that the equation $2n = k^2 - m^2$ has at least 2017 solutions in positive integers k, m.

Let us take 2017 distinct Pythagorean triples (r_i, s_i, t_i), $1 \leq i \leq 2017$ (so that $r_i^2 + s_i^2 = t_i^2$), and let $a = \Pi_{i=1}^{2017} s_i$. We may assume that at least one of the s_i's is even, so that a is even. Put $n = a^2/2$. Now, $m_i = ar_i/s_i$ and $k_i = at_i/s_i$ are both integers and $m_i^2 + a^2 = \frac{a^2(r_i^2+s_i^2)}{s_i^2} = k_i^2$ for any $1 \leq i \leq 2017$. This gives 2017 solutions of the equation $k^2 - m^2 = 2n$ and hence at least 2017 perfect pairings of S_n.

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