2017 Canadian Open Mathematics Challenge

Presented by the Canadian Mathematical Society and supported by the Actuarial Profession.

CMS SMC

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SOCIETY OF ACTUARIES

Participant Identification

Please print clearly and complete all information below. Failure to print legibly or provide complete information may result in your exam being disqualified. This exam is not considered valid unless it is accompanied by your proctor's/teacher's signed form.

1. Given/First Name: (Required)

2. Sur-/Last Name: (Required)

3. Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since September 15th, 2017? (Required for qualification) ☐ **Yes** ☐ No

4. Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of your current address)? (Required for qualification) ☐ **Yes** ☐ No

5. Date of Birth: (Required)

Y Y Y Y M M D D

6. Grade: (Required)

☐ 8 ☐ 10 ☐ 12
☐ 9 ☐ 11 ☐ Cégep

other: _____________

7. Gender: (Optional)

☐ Female
☐ Male

8. Email Address: (Optional)

UBC

Special thanks to the ***University of British Columbia****, marking partner for COMC 2017!*

DO NOT PHOTOCOPY BLANK EXAMS!
Each page of each copy is uniquely pre-coded to facilitate computer-assisted marking.

Question A1 (4 points)

Uniquely-identified page
NO PHOTOCOPIES!

The average of the numbers $2, 5, x, 14, 15$ is x. Determine the value of x.

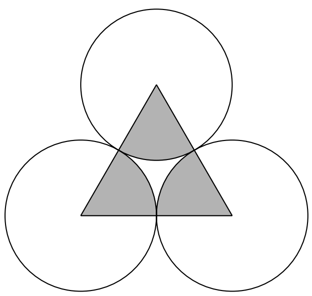
Your solution:

Your final answer:

Question A2 (4 points)

Uniquely-identified page
NO PHOTOCOPIES!

An equilateral triangle has sides of length 4cm. At each vertex, a circle with radius 2cm is drawn, as shown in the figure below. The total area of the shaded regions of the three circles is $a \times \pi$ cm^2. Determine a.

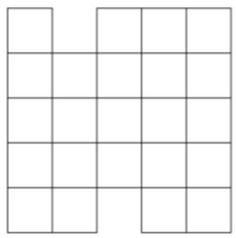


Your solution:

Your final answer:

Question A3 (4 points)

Two 1×1 squares are removed from a 5×5 grid as shown.



Determine the total number of squares of various sizes on the grid.

Your solution:

Your final answer:

Question A4 (4 points)

Three positive integers a, b, c satisfy

$$4^a \times 5^b \times 6^c = 8^8 \times 9^9 \times 10^{10}.$$

Determine the sum of $a + b + c$.

Your solution:

Your final answer:

Question B1 (6 points)

Andrew and Beatrice practice their free throws in basketball. One day, they attempted a total of 105 free throws between them, with each person taking at least one free throw. If Andrew made exactly 1/3 of his free throw attempts and Beatrice made exactly 3/5 of her free throw attempts, what is the highest number of successful free throws they could have made between them?

Your solution:

Your final answer:

Question B2 (6 points)

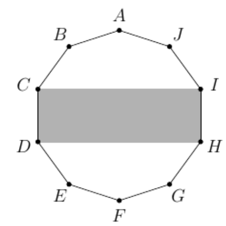
There are twenty people in a room, with a men and b women. Each pair of men shakes hands, and each pair of women shakes hands, but there are no handshakes between a man and a woman. The total number of handshakes is 106. Determine the value of $a \times b$.

Your solution:

Your final answer:

Question B3 (6 points)

Regular decagon (10-sided polygon) $ABCDEFGHIJ$ has area 2017 square units. Determine the area (in square units) of the rectangle $CDHI$.



Your solution:

Your final answer:

Question B4 (6 points)

Numbers a, b and c form an arithmetic sequence if $b - a = c - b$. Let a, b, c be positive integers forming an arithmetic sequence with $a < b < c$. Let $f(x) = ax^2 + bx + c$. Two distinct real numbers r and s satisfy $f(r) = s$ and $f(s) = r$. If $rs = 2017$, determine the smallest possible value of a.

Your solution:

Your final answer:

Question C1 (10 points)

For a positive integer n, we define function $P(n)$ to be the sum of the digits of n **plus** the number of digits of n. For example, $P(45) = 4 + 5 + 2 = 11$. (Note that the first digit of n reading from left to right, cannot be 0).

(a) Determine $P(2017)$.

(b) Determine all numbers n such that $P(n) = 4$.

(c) Determine with an explanation whether there exists a number n for which $P(n) - P(n + 1) > 50$.

Your solution:

Question C2 (10 points)

A function $f(x)$ is periodic with period $T > 0$ if $f(x+T) = f(x)$ for all x. The smallest such number T is called the least period. For example, the functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π.

(a) Let a function $g(x)$ be periodic with the least period $T = \pi$. Determine the least period of $g(x/3)$.

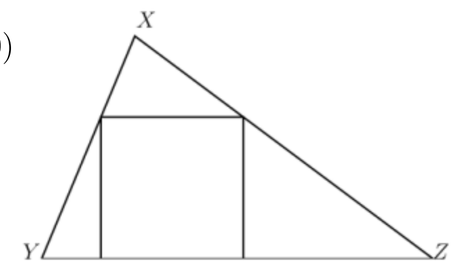
(b) Determine the least period of $H(x) = \sin(8x) + \cos(4x)$

(c) Determine the least periods of each of $G(x) = \sin(\cos(x))$ and $F(x) = \cos(\sin(x))$.

Your solution:

Question C3 (10 points)

Let XYZ be an acute-angled triangle. Let s be the side-length of the square which has two adjacent vertices on side YZ, one vertex on side XY and one vertex on side XZ. Let h be the distance from X to the side YZ and let b be the distance from Y to Z.



(a) If the vertices have coordinates $X = (2, 4)$, $Y = (0, 0)$ and $Z = (4, 0)$, find b, h and s.

(b) Given the height $h = 3$ and $s = 2$, find the base b.

(c) If the area of the square is 2017, determine the minimum area of triangle XYZ.

Your solution:

Question C4 (10 points)

Let n be a positive integer and $S_n = \{1, 2, \ldots, 2n-1, 2n\}$. A *perfect pairing* of S_n is defined to be a partitioning of the $2n$ numbers into n pairs, such that the sum of the two numbers in each pair is a perfect square. For example, if $n = 4$, then a perfect pairing of S_4 is $(1, 8), (2, 7), (3, 6), (4, 5)$. It is not necessary for each pair to sum to the same perfect square.

(a) Show that S_8 has at least one perfect pairing.

(b) Show that S_5 does not have any perfect pairings.

(c) Prove or disprove: there exists a positive integer n for which S_n has at least 2017 different perfect pairings. (Two pairings that are comprised of the same pairs written in a different order are considered the same pairing.)

Your solution:

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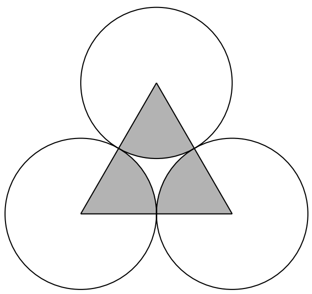
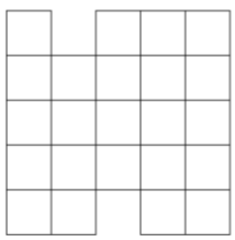
2017 COMC Problem Set (Short Form)

A1. The average of the numbers $2, 5, x, 14, 15$ is x. Determine the value of x.

A2. An equilateral triangle has sides of length 4cm. At each vertex, a circle with radius 2cm is drawn, as shown in the figure below. The total area of the shaded regions of the three circles is $a \times \pi$ cm^2. Determine a.



A3. Two 1×1 squares are removed from a 5×5 grid as shown.



Determine the total number of squares of various sizes on the grid.

A4. Three positive integers a, b, c satisfy

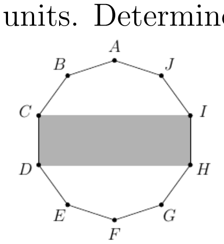
$$4^a \times 5^b \times 6^c = 8^8 \times 9^9 \times 10^{10}.$$

Determine the sum of $a + b + c$.

B1. Andrew and Beatrice practice their free throws in basketball. One day, they attempted a total of 105 free throws between them, with each person taking at least one free throw. If Andrew made exactly 1/3 of his free throw attempts and Beatrice made exactly 3/5 of her free throw attempts, what is the highest number of successful free throws they could have made between them?

B2. There are twenty people in a room, with a men and b women. Each pair of men shakes hands, and each pair of women shakes hands, but there are no handshakes between a man and a woman. The total number of handshakes is 106. Determine the value of $a \times b$.

B3. Regular decagon (10-sided polygon) $ABCDEFGHIJ$ has area 2017 square units. Determine the area (in square units) of the rectangle $CDHI$.



B4. Numbers a, b and c form an arithmetic sequence if $b - a = c - b$. Let a, b, c be positive integers forming an arithmetic sequence with $a < b < c$. Let $f(x) = ax^2 + bx + c$. Two distinct real numbers r and s satisfy $f(r) = s$ and $f(s) = r$. If $rs = 2017$, determine the smallest possible value of a.

C1. For a positive integer n, we define function $P(n)$ to be the sum of the digits of n **plus** the number of digits of n. For example, $P(45) = 4 + 5 + 2 = 11$. (Note that the first digit of n reading from left to right, cannot be 0).

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(b) Determine all numbers n such that $P(n) = 4$.

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C2. A function $f(x)$ is periodic with period $T > 0$ if $f(x+T) = f(x)$ for all x. The smallest such number T is called the least period. For example, the functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π.

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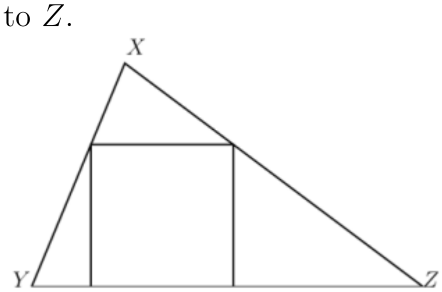
C3. Let XYZ be an acute-angled triangle. Let s be the side-length of the square which has two adjacent vertices on side YZ, one vertex on side XY and one vertex on side XZ. Let h be the distance from X to the side YZ and let b be the distance from Y to Z.



(a) If the vertices have coordinates $X = (2, 4)$, $Y = (0, 0)$ and $Z = (4, 0)$, find b, h and s.

(b) Given the height $h = 3$ and $s = 2$, find the base b.

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C4. Let n be a positive integer and $S_n = \{1, 2, \ldots, 2n-1, 2n\}$. A *perfect pairing* of S_n is defined to be a partitioning of the $2n$ numbers into n pairs, such that the sum of the two numbers in each pair is a perfect square. For example, if $n = 4$, then a perfect pairing of S_4 is $(1, 8), (2, 7), (3, 6), (4, 5)$. It is not necessary for each pair to sum to the same perfect square.

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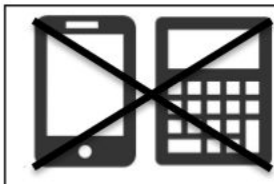
The 2017 Canadian Open Mathematics Challenge

November 2/3, 2017

STUDENT INSTRUCTIONS

General Instructions:



Mobile phones and calculators are NOT permitted.

1) Do not open the exam booklet until instructed to do so by your proctor (supervising teacher).

2) **Before the exam time starts, the proctor will give you a few minutes to fill in the Participant Identification on the cover page of the exam.** You don't need to rush. Be sure to fill in all required information fields and write legibly.

3) **Readability counts:** Make sure the pencil(s) you use are dark enough to be clearly legible throughout your exam solutions.

4) Once you have completed the exam and given it to the proctor/teacher you may leave the room.

5) The questions and solutions of the COMC exam <u>must not</u> be publicly discussed or shared (including online) for at least 24 hours.

Exam Format:

There are three parts to the COMC to be completed in a total of 2 hours and 30 minutes:

PART A: Four introductory questions worth 4 marks each. <u>You do not have to show your work.</u> A correct final answer gives full marks. However, if your final answer is incorrect and you have shown your work in the space provided, you might earn partial marks.

PART B: Four more challenging questions worth 6 marks each. Marking and partial marks follow the same rule as part A.

PART C: Four long-form proof problems worth 10 marks each. <u>Complete work must be shown.</u> Partial marks may be awarded.

Diagrams provided are *not* drawn to scale; they are intended as aids only.

Scrap paper/extra pages: You *may* use scrap paper, but you have to throw it away when you finish your work and hand in your booklet. <u>Only the work you do on the pages provided in the booklet will be evaluated for marking.</u> Extra pages are not permitted to be inserted in your booklet.

Exact solutions: It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., rather than as 12.566, 4.646, etc.

Awards: The names of all award winners will be published on the Canadian Mathematical Society web site.