# Canadian Open Mathematics Challenge 2016

Sun Life Financial

**Official Solutions**

COMC exams from other years, with or without the solutions included, are free to download online. Please visit http://comc.math.ca/2016/practice.html

The COMC has 3 sections:

**A.** Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

**B.** Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

**C.** Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

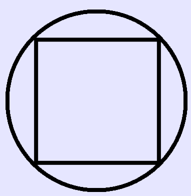
 

# Section A – 4 marks each

1. Pat has ten tests to write in the school year. He can obtain a maximum score of 100 on each test. The average score of Pat's first eight tests is 80 and the average score of all of Pat's tests is $N$. What is the maximum possible value of $N$?

**Solution:** Pat's average will be highest if he scores as high as possible on his last two tests and the highest he can score on each test is 100. His total score for the ten tests will be $8 * 80 + 100 + 100 = 840$ and his average will be $840/10 = 84$.

2. A square is inscribed in a circle, as shown in the figure. If the area of the circle is $16\pi$ cm$^2$ and the area of the square is $S$ cm$^2$, what is the value of S?



**Solution:** The area of a circle with radius $r$ is $\pi r^2$. Since this circle has area $16\pi$, we have $r^2 = 16$, and $r = 4$. So the circle has radius 4 and diameter 8.

Let $t$ be the side length of the square. By the Pythagorean Theorem, $t^2 + t^2 = 8^2$ so $t^2 = 32$, which is the area of the square.

3. Determine the pair of real numbers $x, y$ which satisfy the system of equations:

$$\begin{aligned} \frac{1}{x}+\frac{1}{y} &= 1 \\ \frac{2}{x}+\frac{3}{y} &= 4 \end{aligned}$$

**Solution 1:** Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$. We can rewrite our equations as:

$$\begin{aligned} a+b &= 1 \\ 2a+3b &= 4 \end{aligned}$$

Subtracting twice the first equation from the second equation gives $b = 2$. Substituting this into one of the equations gives $a = -1$. These give $x = -1$ and $y = \frac{1}{2}$.

**Solution 2:** Taking 3 times the first equation and subtracting the second equation gives $\frac{1}{x} = -1$, so $x = -1$. Substituting this into the first equation gives $-1 + \frac{1}{y} = 1$ which yields $y = \frac{1}{2}$. Thus $(x, y) = (-1, \frac{1}{2})$.

4. Three males and two females write their names on sheets of paper, and randomly arrange them in order, from left to right. What is the probability that all of the female names appear to the right of all the male names?

**Solution 1:** We need the first (leftmost) three names to be males and the last two names to be females. The probability that the fifth name is female is $\frac{2}{5}$. Given that the fifth name is female, the probability that the fourth name is also female is $\frac{1}{4}$. The probability that both are female is therefore $\frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$.

**Solution 2:** We need the first (leftmost) three names to be males and the last two names to be females. There are $\binom{5}{2} = 10$ choices for which two names are the female names, so the probability the last two names are females is $\frac{1}{10}$.

# Section B – 6 marks each

1. If the cubic equation $x^3 - 10x^2 + Px - 30 = 0$ has three positive integer roots, determine the value of $P$.

**Solution 1:** If the equation has roots $x_1, x_2, x_3$, then $x_1+x_2+x_3 = 10, x_1x_2+x_2x_3+x_3x_1 = P$ and $x_1x_2x_3 = 30$. Suppose $x_1 = 1$ is a root. Then $x_2 + x_3 = 9$ and $x_2x_3 = 30$. For any pair of integers which multiply to 30, their sum is at least 11, so this is not possible, and 1 is not a root. The only other possibility is that the roots are $2, 3, 5$, which gives $P = 2 \cdot 3 + 3 \cdot 5 + 5 \cdot 2 = 31$.

**Solution 2:** The roots of the equation are integers and have a product of 30. The possible sets of roots are $(1, 1, 30), (1, 2, 15), (1, 3, 10), (1, 5, 6), (2, 3, 5)$. Of these, only $(2, 3, 5)$ sums to 10, so it must be our set of roots. To find P, we substitute $x = 2$ into the equation to get $8 - 40 + 2P - 30 = 0$, which yields $P = 31$.

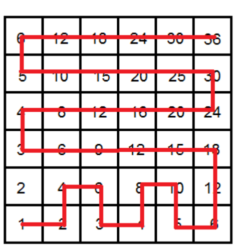
2. The squares of a $6 \times 6$ square grid are each labeled with a *point value*. As shown in the diagram below, the point value of the square in row $i$ and column $j$ is $i \times j$.

| | | | | | |
|---|---|---|---|---|---|
| 6 | 12 | 18 | 24 | 30 | 36 |
| 5 | 10 | 15 | 20 | 25 | 30 |
| 4 | 8 | 12 | 16 | 20 | 24 |
| 3 | 6 | 9 | 12 | 15 | 18 |
| 2 | 4 | 6 | 8 | 10 | 12 |
| 1 | 2 | 3 | 4 | 5 | 6 |

A *path* in the grid is a sequence of squares, such that consecutive squares share an edge and no square occurs twice in the sequence. The *score* of a path is the sum of the point values of all squares in the path.
Determine the highest possible score of a path that begins with the bottom left corner of the grid and ends with the top right corner.

**Solution:** Let us colour the squares of the grid with a checkerboard pattern of black and white, starting with black in the bottom left-hand corner. A path in the grid will alternate between black and white squares, beginning and ending on black. Thus, any path cannot contain all 18 white squares. The path in the figure below contains all squares except for the white square with the lowest point value (2), so it will have the highest possible score.

The sum of the point values for all the squares is $(1+2+3+4+5+6)(1+2+3+4+5+6) = 441$, and the score of the path is 439.



3. A hexagon $ABCDEF$ has $AB = 18$cm, $BC = 8$cm, $CD = 10$cm, $DE = 15$cm, $EF = 20$cm, $FA = 1$cm, $\angle FAB = 90^\circ$, $\angle CDE = 90^\circ$ and $BC$ is parallel to $EF$. Determine the area of this hexagon, in cm$^2$.

**Solution:** First, we determine the length of $BF$ and $CE$. By the Pythagorean theorem, $BF = \sqrt{AB^2 + AF^2} = \sqrt{18^2 + 1^2} = \sqrt{325}$cm and $CE = \sqrt{CD^2 + DE^2} = \sqrt{10^2 + 15^2} = \sqrt{325}$cm. Furthermore, note that $BF = CE$.

Since $BC$ is parallel to $CF$ and $BF = CE$, $BCEF$ is an isosceles trapezoid with $CE = BF$.

The hexagon comprises of two right-angled triangles $FAB$ and $CDE$ and the isosceles trapezoid $BCEF$. The area of $FAB$ is $FA \times AB/2 = 1 \times 18/2 = 9$cm$^2$ and the area of $CDE$ is $CD \times DE/2 = 10 \times 15/2 = 75$cm$^2$. Now, we determine the area of the isosceles trapezoid $BCEF$.

Note that $BC$ is parallel to $EF$ and $BC < EF$ and $BF = CE$. Drop the perpendicular from $B, C$ to side $EF$, touching $EF$ at $X, Y$, respectively. Then $XY = BC = 8$cm, $FX + XY + YE = EF = 20$cm and $FX = YE$, since $BCEF$ is an isosceles trapezoid. Since $FX + YE = 20 - XY = 20 - 8 = 12$cm, $FX = YE = 6$cm. Therefore, by the Pythagorean Theorem, $BX = \sqrt{BF^2 - FX^2} = \sqrt{325 - 36} = \sqrt{289} = 17$cm. Therefore, the height of the trapezoid $BCEF$ is 17cm. Hence, the area of the trapezoid is $(BC + EF) \times BX/2 = (8 + 20) \times 17/2 = 14 \times 17 = 238$cm$^2$.

Summing all three pieces, the area of the hexagon is $9 + 75 + 238 = 322$cm$^2$.

4. Let $n$ be a positive integer. Given a real number $x$, let $\lfloor x \rfloor$ be the greatest integer less than or equal to $x$. For example, $\lfloor 2.4 \rfloor = 2$, $\lfloor 3 \rfloor = 3$ and $\lfloor \pi \rfloor = 3$. Define a sequence $a_1, a_2, a_3, \ldots$ where $a_1 = n$ and

$$a_m = \left\lfloor \frac{a_{m-1}}{3} \right\rfloor,$$

for all integers $m \geq 2$. The sequence stops when it reaches zero. The number $n$ is said to be *lucky* if 0 is the only number in the sequence that is divisible by 3.
For example, 7 is lucky, since $a_1 = 7, a_2 = 2, a_3 = 0$, and none of $7, 2$ are divisible by 3. But 10 is not lucky, since $a_1 = 10, a_2 = 3, a_3 = 1, a_4 = 0$, and $a_2 = 3$ is divisible by 3. Determine the number of lucky positive integers less than or equal to 1000.

**Solution**[1]**:** Note that $\lfloor m/3 \rfloor$ is simply chopping off the right-most digit of the base 3 representation of $m$. Hence, what the sequence is doing is that given an integer $n$, we repeatedly chop the right-most digit of the base 3 representation of $n$, until no digits remain. A base 3 representation is divisible by 3 if and only if the right most digit is zero. Then $n$ is lucky if and only if none of the digits of the base 3 representation of $n$ is zero.

We convert the integer $n$ to base 3. We first find the base 3 representation of 1000. Note that $1000/3 = 333 + 1/3$, $333/3 = 111 + 0/3$, $111/3 = 37 + 0/3$, $37/3 = 12 + 1/3$, $12/3 = 4 + 0/3$, $4/3 = 1 + 1/3$ and $1/3 = 0 + 1/3$. Therefore, the base 3 representation of 1000 is $1101001_3$. We need to find the number of base 3 representations, whose digits does not contain zero, less than or equal to this number.

Note that the largest lucky number less than or equal to $1101001_3$ is is $222222_3$. Lucky numbers less than or equal to 1000 are those whose base 3 representation consists of only digits 1 and 2, and has at most 6 digits. There are $2^k$ such numbers whose base 3 representation has $k$ digits. Therefore, the total number of lucky numbers is $2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 = 126$.

[1] Typos corrected from the unofficial solution. Thanks to Jeff Zhang of Lord Byng Secondary in Vancouver.

# Section C – 10 marks each

1. A sequence of three numbers $a, b, c$ form an arithmetic sequence if the difference between successive terms in the sequence is the same. That is, when $b - a = c - b$.

   (a) The sequence $2, b, 8$ forms an arithmetic sequence. Determine $b$.

   (b) Given a sequence $a, b, c$, let $d_1$ be the non-negative number to increase or decrease $b$ by so that, without changing $a$ or $c$, the result is an arithmetic sequence. Let $d_2$ be the positive number to increase or decrease $c$ by so that, without changing $a$ or $b$, the result is an arithmetic sequence.

   For example, if the three-term sequence is $3, 10, 13$, then we need to decrease 10 to 8 to make the arithmetic sequence $3, 8, 13$. We decreased $b$ by 2, so $d_1 = 2$. If we change the third term, we need to increase 13 to 17 to make the arithmetic sequence $3, 10, 17$. We increased 13 by 4, so $d_2 = 4$.

   Suppose the original three term sequence is $1, 13, 17$. Determine $d_1$ and $d_2$.

   (c) Define $d_1, d_2$ as in part (b). For all three-term sequences, prove that $2d_1 = d_2$.

(a) **Solution 1:** We require that $b - a = c - b$, so $b - 2 = 8 - b$ and $b = 5$.

**Solution 2:** $b$ must be in the middle of $a$ and $c$, so b $= (8 + 2)/2 = 5$.

**Solution 3:** $2, 5, 8$ forms an arithmetic sequence, so $b = 5$.

(b) **Solution 1:** $b - a = c - b$, and $a = 1, c = 17$. Substituting into the equation and solving gives $b = 9$. Since we have 13, as our middle term, the difference must be $d_1 = 13 - 9 = 4$.

Similarly, if we have $a = 1, b = 13$, solving gives $c = 25$. The difference must be $d_2 = 25 - 17 = 8$.

**Solution 2:** The second number must be 9 to form an arithmetic sequence, so $d_1 = 13 - 9 = 4$.

Similarly, the third number must be 25 to form an arithmetic sequence, so $d_2 = 25 - 17 = 8$.

(c) **Solution 1:** We can rewrite the relation for an arithmetic sequence as $2b = a + c$. Suppose we have a sequence (not necessarily arithmetic) $a, b, c$, then we can write $2(b \pm d_1) = a + c$, and we can also write $2b = a + (c \pm d_2)$. Subtracting one equation from the other we are left with $2d_1 = \pm d_2$ and since both $d_1$ and $d_2$ are positive $2d_1 = d_2$.

**Solution 2:** We first determine $d_1$. Let $b'$ be the number we replace $b$ with to form an arithmetic sequence. Then $b' - a = c - b'$. Therefore, $b' = (a + c)/2$. Hence, we changed $b$ by $(a + c)/2 - b$. Hence, $d_1 = |(a + c)/2 - b| = |(a + c - 2b)/2|$, where $| \cdot |$ denotes absolute value. (The contestant shouldn't need to use absolute value. Any explanation that we take the positive version of the number would suffice.)

We next determine $d_2$. Let $c'$ be the number to replace $c$ by to make the sequence arithmetic. Then $c' - b = b - a$. Therefore, $c' = 2b - a$. Hence, we change $c$ by $c' - c = 2b - a - c$. Therefore, $d_2 = |2b - a - c| = 2|(2b - a - c)/2| = 2|(a + c - 2b)/2| = 2d_1$, as desired.

**Solution 3:** Place the points $A(1, a), B(2, b), C(3, c)$ on the coordinate plane. Let $b'$ be the number so that $a, b', c$ is an arithmetic sequence and $c'$ the number so that $a, b, c'$ is an arithmetic sequence. Let $B' = (2, b')$ and $C' = (3, c')$. Then $A, B', C$ lie on a straight line and $A, B, C'$ lie on a straight line. Furthermore, $B'$ is the midpoint of $AC$ and $B$ is the midpoint of $AC'$.

Note that $d_1$ is the distance between $B$ and $B'$ and $d_2$ is the distance between $C$ and $C'$. Consider triangle $ACC'$. Since $BB', CC'$ are both vertical, they are parallel. Therefore, $\Delta ABB'$ and $\Delta AC'C$ are similar, with ratio $1 : 2$. Therefore, $CC'$ is twice the length of $BB'$, yielding $d_2 = 2d_1$.

2. Alice and Bob play a game, taking turns, playing on a row of $n$ seats. On a player's turn, he or she places a coin on any seat provided there is no coin on that seat or on an adjacent seat. Alice moves first. The player who does not have a valid move loses the game.

   (a) Show that Alice has a winning strategy when $n = 5$.

   (b) Show that Alice has a winning strategy when $n = 6$.

   (c) Show that Bob has a winning strategy when $n = 8$.

(a) **Solution:** On her first turn, Alice places a coin on seat 3. Bob can now only place coins on seats 1 or 5. Whichever seat Bob places a coin on, Alice can then place a coin on the other seat. After this, Bob is unable to play and so loses the game.

(b) **Solution:** On her first turn, Alice places the coin on seat 3. Bob can now only place coins on seats 1, 5, or 6. If Bob places the coin on 1, Alice places a coin on 5, and Bob is unable to play on 6, and so loses. If Bob places the coin in 5 or 6, then the other can no longer have a coin placed on it. Alice places a coin on 1, and then Bob has no place to play, and so he loses.

(c) **Solution:** If Alice places her first coin on seat 1 or seat 8, there are now 6 consecutive seats that could have coins places on them. This is equivalent to part (b), except Bob goes first, so he will win this game. Similarly, if Alice places her first coin on seat 2 or 7, there are now 5 consecutive seats that could have coins placed on them, and this is equivalent to (a), except Bob goes first.

If Alice places her first coin in seat 3 or 6, Bob places his first coin in the other. Now, only seats 1 and 8 can have a coin played in them, and whichever one Alice plays on, Bob will play on the other, winning the game.

If Alice places her coin in seat 4, Bob places his coin in 6. This leaves 1, 2, and 8 as places that a coin may be placed. As in part (b), we see that exactly two more turns will occur, meaning Bob wins. If Alice places first coin in 5, Bob plays symmetrically to the aforementioned case.

3. Let $A = (0, a), O = (0, 0), C = (c, 0), B = (c, b)$, where $a, b, c$ are positive integers. Let $P = (p, 0)$ be the point on line segment $OC$ that minimizes the distance $AP + PB$, over all choices of $P$. Let $X = AP + PB$.

   (a) Show that this minimum distance is $X = \sqrt{c^2 + (a + b)^2}$

   (b) If $c = 12$, find all pairs $(a, b)$ for which $a, b, p$, and $X$ are positive integers.

   (c) If $a, b, p, X$ are all positive integers, prove that there exists an integer $n \geq 3$ that divides both $a$ and $b$.

(a) **Solution:** Reflect the point $B$ in the horizontal x-axis to get the point $B'$. Then $PB = PB'$. By the triangle inequality, $PA + PB' \geq AB'$. The line $AB'$ will pass through the segment $OC$, so taking $P$ along $AB'$ is where $PA + PB'$ will attain its minimum.

By the Pythagorean Theorem, the length of $AB'$ is $X = \sqrt{c^2 + (a + b)^2}$.

(b) **Solution:** Observe that at the minimum point attained in (a), $POA$ and $PCB$ are similar triangles. Let $a + b = n$. We have $X^2 = 144 + n^2$, or $(X - n)(X + n) = 144$. Since $X$ and $n$ are both integers, we have $X - n$ and $X + n$ are the same parity. Since their product is a multiple of 2, each must also be a multiple of 2. We can factor 144 in the following ways, where both factors are even: $(72, 2), (36, 4), (18, 8), (12, 12), (24, 6)$. These yield $(X, n)$ pairs of $(37, 35), (20, 16), (13, 5), (12, 0), (15, 9)$. We can immediately eliminate $(12, 0)$, as $n$ must be positive.

Notice that by similar triangles, $p/12 = a/n$, so $p = 12a/n$. If $n = 5, 35$, there is no value of $a$ for which $b$ will also be positive.

When $n = 9$, we have $(a, b) = (3, 6), (6, 3)$ and when $n = 16$, we have $(a, b) = (4, 12), (8, 8), (12, 4)$.

(c) **Solution**[2]**:** Let $q = c - p$. By similar triangles, $p = ac/(a + b)$ and $q = bc/(a + b)$.

Let $k$ be the smallest integer such that $kc$ is divisible by $a + b$. Then since $p$ and $q$ are integers, k must divide $a$ and $b$.

If $k = 1$, then $c = m(a + b)$ for some integer $m$, and $X = \sqrt{1 + m^2}(a + b)$. This is only an integer when m = 0, which it cannot be, since then c = 0. This $k > 1$.

If $k = 2$, then $2c = m(a + b)$ for some integer $m$, and $X = (\frac{a+b}{2})\sqrt{4 + m^2}$. Again, this is only an integer when $m = 0$, so $k > 2$, as required.

[2]Typo corrected from the unofficial solutions. Thanks to Jeff Zhang of Lord Byng Secondary in Vancouver.

4. Two lines intersect at a point $Q$ at an angle $\theta^\circ$, where $0 < \theta < 180$. A frog is originally at a point other than $Q$ on the angle bisector of this angle. The frog alternately jumps over these two lines, where a jump over a line results in the frog landing at a point which is the reflection across the line of the frog's jumping point.
The frog stops when it lands on one of the two lines.

   (a) Suppose $\theta = 90^\circ$. Show that the frog never stops.

   (b) Suppose $\theta = 72^\circ$. Show that the frog eventually stops.

   (c) Determine the number of integer values of $\theta$, with $0 < \theta^\circ < 180^\circ$, for which the frog never stops.

(a) **Solution:** Suppose we orient the lines so that $\ell_1$ is the $x$-axis and $\ell_2$ is the $y$-axis. Again, suppose that the frog jumps over $\ell_1$ first. If the frog begins at a point $(r, r)$, where $r > 0$. Then the subsequent points of the frog are $(r, -r), (-r, -r), (-r, r), (r, r)$ and back to $(r, -r)$. The frog then repeats these four locations indefinitely. Hence, the frog never stops.

(b) **Solution:** Let $\ell_1, \ell_2$ be the two lines, and let $\ell$ be the angle bisector of the $72^\circ$ angle. The frog is originally on a point $F$ on $\ell$, not on point $O$. By symmetry, we can assume that the frog is jumping over $\ell_1$. Note that the lines $\ell$ and $\ell_1$ intersect at $36^\circ$. Let $F'$ be the point of reflection of $F$ across $\ell_1$. Then $OF'$ and $\ell_1$ intersect at $36^\circ$. Consequently, $OF'$ and $\ell_2$ intersect at $108^\circ$. Note that the external (obtuse) angle formed by $\ell_1$ and $\ell_2$ is equal to $180 - 72 = 108^\circ$. Hence, on the frog's next jump, the frog will land on $\ell_1$ and the frog stops.

(c) **Solution**[3] **:** We will show that the frog stops if and only if $\theta$ is a multiple of 8. From this, we see that there are $\lfloor 179/8 \rfloor = 22$ multiples of 8 from 1 to 179 (inclusively). Therefore, there are $179 - 22 = 157$ integer values of $\theta$ for which the frog never stops.

Let $O$ be the origin on the coordinate plane, with $\ell_1$ coincide with the $x$-axis, and $\ell_2$ is placed so that it forms the angle $\theta$ with the positive $x$-axis of $\ell_1$.

The first key observation is that the frog is always the same distance from $O$ after it jumps. This is because the two lines intersect at $O$, and reflection across these two lines preserve the distance of the frog to $O$, since distances are preserved on reflection.

Hence, we can measure the size of the directed angle, in the counter-clockwise direction, formed by the ray starting at $O$ toward the positive $x$-axis (i.e. of $\ell_1$) to the ray $OF$, where $F$ is the location of the frog. We will use the size of this angle to measure where the frog is. Since a full circle is $360^\circ$, angles that are equal modulo $360^\circ$, are equal.[4]

[3]Typos corrected from the unofficial version. Thanks to Jeff Zhang of Lord Byng Secondary in Vancouver.
[4]Yes, these are polar coordinates. But I'm trying not to use this terminology in the solution.

There are four locations on which the frog can land on the line, particular at angles $0, \theta, 180$ and $180+\theta$. If the frog is forced to land in one of these four positions, then the frog will stop.

Given a location $\alpha$, let $f(\alpha)$ be the location of the frog after jumping across $\ell_1$ and $g(\alpha)$ be the location of the frog after jumping across $\ell_2$.

We first determine explicit formulae for $f(\alpha)$ and $g(\alpha)$. $f(\alpha)$ is simply a reflection about the $x$-axis. Therefore, $f(\alpha) = -\alpha$, where again we consider these values modulo 360. $g(\alpha)$ is a reflection about the line $\ell_2$, which passes through the location $\theta$. To determine $g(\alpha)$, we rotate the line clockwise by $\theta$, perform the reflection, and re-rotate the line counterclockwise by $\theta$. Rotating the line clockwise by $\theta$ changes $\alpha$ to $\alpha - \theta$. Reflecting across the $x$-axis yields $\theta - \alpha$. Then rotating back counterclockwise by $\theta^\circ$ yields $2\theta - \alpha$. Therefore, $g(\alpha) = 2\theta - \alpha$.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be the sequence of locations of the frog after each jump, where $\alpha_1$ is the frog's original location, which is $\theta/2$. Without loss of generality, suppose the frog jumps across $\ell_1$ first. Then $\alpha_2 = f(\alpha_1) = -\alpha_1 = -\theta/2$. Furthermore, $\alpha_3 = g(\alpha_2) = g(f(\alpha_1))$ and $\alpha_4 = f(\alpha_3) = f(g(\alpha_2))$. Generally, $\alpha_{n+2} = g(f(\alpha_n))$ for all odd $n$ and $\alpha_{n+2} = f(g(\alpha_n))$ for all even $n$.

Note that $f(g(\alpha)) = f(2\theta - \alpha) = \alpha - 2\theta$ and $g(f(\alpha)) = g(-\alpha) = 2\theta + \alpha$.

Therefore, $\alpha_1 = \theta/2, \alpha_3 = 2\theta + \theta/2.\alpha_5 = 4\theta + \theta/2$. In general, $\alpha_{2k+1} = 2k\theta + \theta/2$ for all non-negative integers $k$. In the even case, $\alpha_2 = -\theta_2.\alpha_4 = -2\theta - \theta/2, \alpha_6 = -4\theta - \theta/2$. In general, $\alpha_{2k+2} = -2k\theta - \theta/2$ for all non-negative integers $k$.

Now, the frog will stop if there exists a non-negative integer $k$ where either $2k\theta + \theta/2$ or $-2k\theta - \theta/2$ is equal to any one of $0, \theta, 180, 180+\theta$, modulo 360. This is equivalent to $2k\theta + \theta/2$ or $-2k\theta - \theta/2$ is equal to any one of 0 or $\theta$ modulo 180.

Note that $2k\theta + \theta/2 \equiv 0 \pmod{180}$ if and only if $(4k+1)\theta \equiv 0 \pmod{360}$. Since $4k+1$ is odd and 360 is divisible by 8, $\theta$ must be divisible by 8 for there to be a solution. Conversely, if $\theta$ is divisible by 8, then setting $k = 11$ yields a value of $45\theta$ on the left side, which is indeed divisible by 360.

Note that $2k\theta + \theta/2 \equiv \theta \pmod{180}$ if and only if $(4k-1)\theta \equiv 0 \pmod{360}$. Since $4k-1$ is odd and 360 is divisible by 8, $\theta$ must be divisible by 8 for there to be a solution. Conversely, if $\theta$ is divisible by 8, then setting $k = 34$ yields a value of $135\theta = 3 \cdot 45\theta$ on the left hand side, and $45\theta$ would be divisible by $360^\circ$.

Note that $-2k\theta - \theta/2 \equiv 0 \pmod{180}$ if and only if $(4k+1)\theta \equiv 0 \pmod{360}$. This is the same situation as in the first case.

Note that $-2k\theta - \theta/2 \equiv \theta \pmod{180}$ if and only if $(4k+3)\theta \equiv 0 \pmod{360}$. Since $4k+3$ is odd and 360 is divisible by 8, $\theta$ must be divisible by 8 for there to be a solution. Conversely, if $\theta$ is divisible by 8, then setting $k = 33$ yields a value of $(4k+3)\theta = 135\theta = 3 \cdot 45\theta$ on the left side, which is divisible by 360.

Hence, in all cases, there exists a non-negative integer $k$ such that one of $\alpha_{2k+1}, \alpha_{2k+2}$ lies on $\ell_1, \ell_2$, which only occurs if and only if $\theta$ is divisible by 8. This completes the problem.