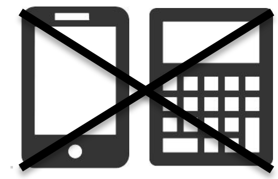
The Canadian Open Mathematics Challenge
November 3/4, 2016

STUDENT INSTRUCTION SHEET

General Instructions



Mobile phones and calculators are NOT permitted.

1) Do not open the exam booklet until instructed to do so by your supervising teacher.
2) The supervisor will give you **five minutes before the exam starts** to fill in the identification section on the exam cover sheet. You don't need to rush. Be sure to fill in all information fields and print legibly.
3) Once you have completed the exam and given it to your supervising teacher you may leave the exam room.
4) The contents of the COMC 2016 exam and your answers and solutions must not be publicly discussed (including online) for at least 24 hours.

Exam Format

You have 2 hours and 30 minutes to complete the COMC. There are three sections to the exam:

PART A: Four introductory questions worth 4 marks each. Partial marks may be awarded for work shown.

PART B: Four more challenging questions worth 6 marks each. Partial marks may be awarded for work shown.

PART C: Four long-form proof problems worth 10 marks each. Complete work must be shown. Partial marks may be awarded.

Diagrams are not drawn to scale; they are intended as aids only.

Work and Answers

All solution work and answers are to be presented in this booklet in the boxes provided – do not include additional sheets. Marks are awarded for completeness and clarity. For sections A and B, it is not necessary to show your work in order to receive full marks. However, if your answer or solution is incorrect, any work that you do and present in this booklet will be considered for partial marks. For section C, you must **show** your work and provide the correct answer or solution to receive full marks.

It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2 + \sqrt{7}$, etc., rather than as 12.566, 4.646, etc. The names of all award winners will be published on the Canadian Mathematical Society web site https://cms.math.ca/comc.

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Please print clearly and ***complete all information below****. Failure to print legibly or provide complete information may result in your exam being disqualified. This exam is not considered valid unless it is accompanied by your test supervisor's signed form.*

First Name:

Last Name:

Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since September 15th of this year?

☐ Yes ☐ No

Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of current address)?

☐ Yes ☐ No

Grade:

☐ 8 ☐ 9 ☐ 10

☐ 11 ☐ 12 ☐ Cégep

Other: ________

T-Shirt Size:

☐ XS ☐ S ☐ M

☐ L ☐ XL ☐ XXL

Date of Birth:

y y y y m m d d

Gender: (Optional)

☐ Male ☐ Female

E-mail Address:

Signature: ______________________________

For official use only:

A1	A2	A3	A4	B1	B2	B3	B4	C1	C2	C3	C4	TOTAL

_______ Marker initials _______ Data entry initials _______ Vertification Initials

Part A: Question 1 (4 marks)

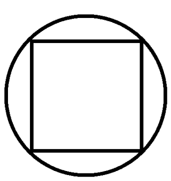
Pat has ten tests to write in the school year. He can obtain a maximum score of 100 on each test. The average score of Pat's first eight tests is 80 and the average score of all of Pat's tests is N. What is the maximum possible value of N?

Your solution:

Your final answer:

Part A: Question 2 (4 marks)

A square is inscribed in a circle, as shown in the figure. If the area of the circle is 16π cm^2 and the area of the square is S cm^2, what is the value of S?



Your solution:

Your final answer:

Part A: Question 3 (4 marks)

Determine the pair of real numbers x, y which satisfy the system of equations:

$$\begin{aligned} \tfrac{1}{x} + \tfrac{1}{y} &= 1 \\ \tfrac{2}{x} + \tfrac{3}{y} &= 4 \end{aligned}$$

Your solution:

Your final answer:

Part A: Question 4 (4 marks)

Three males and two females write their names on sheets of paper, and randomly arrange them in order, from left to right. What is the probability that all of the female names appear to the right of all the male names?

Your solution:

Your final answer:

Part B: Question 1 (6 marks)

If the cubic equation $x^3 - 10x^2 + Px - 30 = 0$ has three positive integer roots, determine the value of P.

Your solution:

Your final answer:

Part B: Question 2 (6 marks)

The squares of a 6×6 square grid are each labelled with a *point value*. As shown in the diagram below, the point value of the square in row i and column j is $i \times j$.

6	12	18	24	30	36
5	10	15	20	25	30
4	8	12	16	20	24
3	6	9	12	15	18
2	4	6	8	10	12
1	2	3	4	5	6

A *path* in the grid is a sequence of squares, such that consecutive squares share an edge and no square occurs twice in the sequence. The *score* of a path is the sum of the point values of all squares in the path.

Determine the highest possible score of a path that begins with the bottom left corner of the grid and ends with the top right corner.

Your solution:

Your final answer:

Part B: Question 3 (6 marks)

A hexagon $ABCDEF$ has $AB = 18$cm, $BC = 8$cm, $CD = 10$cm, $DE = 15$cm, $EF = 20$cm, $FA = 1$cm, $\angle FAB = 90^\circ$, $\angle CDE = 90^\circ$ and BC is parallel to EF. Determine the area of this hexagon, in cm^2.

Your solution:

Your final answer:

Part B: Question 4 (6 marks)

Let n be a positive integer. Given a real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. For example, $\lfloor 2.4 \rfloor = 2$, $\lfloor 3 \rfloor = 3$ and $\lfloor \pi \rfloor = 3$. Define a sequence $a_1, a_2, a_3, \ldots$ where $a_1 = n$ and

$$a_m = \left\lfloor \frac{a_{m-1}}{3} \right\rfloor,$$

for all integers $m \geq 2$. The sequence stops when it reaches zero. The number n is said to be *lucky* if 0 is the only number in the sequence that is divisible by 3.

For example, 7 is lucky, since $a_1 = 7$, $a_2 = 2$, $a_3 = 0$, and none of 7, 2 are divisible by 3. But 10 is not lucky, since $a_1 = 10$, $a_2 = 3$, $a_3 = 1$, $a_4 = 0$, and $a_2 = 3$ is divisible by 3. Determine the number of lucky positive integers less than or equal to 1000.

Your solution:

Your final answer:

Part C: Question 1 (10 marks)

A sequence of three numbers a, b, c form an arithmetic sequence if the difference between successive terms in the sequence is the same. That is, when $b - a = c - b$.

(a) The sequence $2, b, 8$ forms an arithmetic sequence. Determine b.

(b) Given a sequence a, b, c, let d_1 be the non-negative number to increase or decrease b by so that the result is an arithmetic sequence and let d_2 be the positive number to increase or decrease c by so that the result is an arithmetic sequence.

For example, if the three-term sequence is $3, 10, 13$, then we need to decrease 10 to 8 to make the arithmetic sequence $3, 8, 13$. We decreased b by 2, so $d_1 = 2$. If we change the third term, we need to increase 13 to 17 to make the arithmetic sequence $3, 10, 17$. We increased 13 by 4, so $d_2 = 4$.

Suppose the original three term sequence is $1, 13, 17$. Determine d_1 and d_2.

(c) Define d_1, d_2 as in part (b). For all three-term sequences, prove that $2d_1 = d_2$.

Your solution:

Part C: Question 2 (10 marks)

Alice and Bob play a game, taking turns, playing on a row of n seats. On a player's turn, he or she places a coin on any seat provided there is no coin on that seat or on an adjacent seat. Alice moves first. The player who does not have a valid move loses the game.

(a) Show that Alice has a winning strategy when $n = 5$.

(b) Show that Alice has a winning strategy when $n = 6$.

(c) Show that Bob has a winning strategy when $n = 8$.

Your solution:

Part C: Question 3 (10 marks)

Let $A = (0,a), O = (0,0), C = (c,0), B = (c,b)$, where a,b,c are positive integers. Let $P = (p,0)$ be the point on line segment OC that minimizes the distance $AP + PB$, over all choices of P. Let $X = AP + PB$.

(a) Show that this minimum distance is $X = \sqrt{c^2 + (a+b)^2}$

(b) If $c = 12$, find all pairs (a,b) for which a,b,p, and X are positive integers.

(c) If a,b,p,X are all positive integers, prove that there exists an integer $n \geq 3$ that divides both a and b.

Your solution:

Part C: Question 4 (10 marks)

Two lines intersect at a point Q at an angle $\theta°$, where $0 < \theta < 180$. A frog is originally at a point other than Q on the angle bisector of this angle. The frog alternately jumps over these two lines, where a jump over a line results in the frog landing at a point which is the reflection across the line of the frog's jumping point.

The frog stops when it lands on one of the two lines.

(a) Suppose $\theta = 90°$. Show that the frog never stops.

(b) Suppose $\theta = 72°$. Show that the frog eventually stops.

(c) Determine the number of integer values of θ, with $0 < \theta° < 180°$, for which the frog never stops.

Your solution:

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2016 Canadian Open Mathematics Challenge

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