Sun Life Financial Canadian Open Mathematics Challenge 2015

Official Solutions

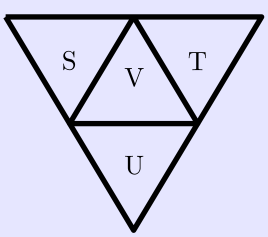
COMC exams from other years, with or without the solutions included, are free to download online. Please visit http://comc.math.ca/2015/practice.html

Section A – 4 marks each

1. A palindrome is a number where the digits read the same forwards or backwards, such as 4774 or 505. What is the smallest palindrome that is larger than 2015?

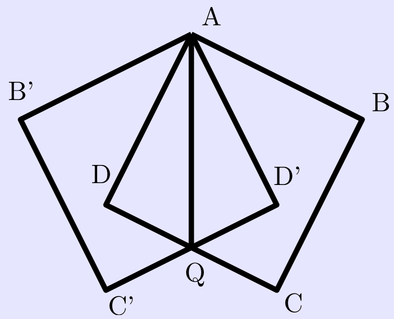
Solution: Since 2015 has 4 digits, any number larger than 2015 will also have 4 digits. Given two starting digits, in order, there is a unique 4-digit palindrome that begins with those two starting digits. For starting digits 20, the palindrome is 2002, which is less than 2015. The next smallest will start with 21 and is 2112. Since $2112 > 2015$ this is the smallest palindrome greater than 2015.

2. In the picture below, there are four triangles labeled S, T, U, and V. Two of the triangles will be coloured red and the other two triangles will be coloured blue. How many ways can the triangles be coloured such that the two blue triangles have a common side?



Solution: For the two blue triangles to have a common side, V must be one of the blue triangles. The other blue triangle could be S, T, or U, so there are 3 different ways to colour the triangles.

3. In the given figure, $ABCD$ is a square with sides of length 4, and Q is the midpoint of CD. $ABCD$ is reflected along the line AQ to give the square $AB'C'D'$. The two squares overlap in the quadrilateral $ADQD'$. Determine the area of quadrilateral $ADQD'$.



Solution: Since the side length of $ABCD$ is 4, so $DQ = 2$. The area of ADQ is $4 \times 2/2 = 4$ and by symmety the area of $AD'Q$ is the same. Thus, the area of $ADQD'$ is $4 + 4 = 8$.

4. The area of a rectangle is 180 units^2 and the perimeter is 54 units. If the length of each side of the rectangle is increased by six units, what is the area of the resulting rectangle?

Solution 1: Let x and y be the sides of the rectangle. Then $xy = 180$ and $2x + 2y = 54$ or $x + y = 27$. By inspection we can see that $x = 12, y = 15$ and $x = 15, y = 12$ are the solutions. When each side is increased by 6, the area of the new rectangle is $(12+6)(15+6) = 18 * 21 = 378$.

Solution 2: Let x and y be the sides of the rectangle. From the question we know that $xy = 180$ and $2x + 2y = 54$. We want to determine

$$\begin{aligned}(x+6)(y+6) &= xy + 6x + 6y + 36 \\ &= xy + 3(2x + 2y) + 36 \\ &= 180 + 162 + 36 \\ &= 378.\end{aligned}$$

Section B – 6 marks each

1. Given an integer $n \geq 2$, let $f(n)$ be the second largest positive divisor of n. For example, $f(12) = 6$ and $f(13) = 1$. Determine the largest positive integer n such that $f(n) = 35$.

Solution: For a positive integer n, let n' be the smallest prime that divides n. Then we can see that $f(n) = n/n'$ and also that $n = f(n) \cdot n'$. We are given that $f(n) = 35$, so to maximize n we must maximize n'. Since n' is the smallest prime factor of n, it cannot be larger than any factor of $f(n)$. Thus, the largest possible value of n' is 5, so the largest possible value of n is $5 \cdot 35 = 175$.

2. Let ABC be a right triangle with $\angle BCA = 90^o$. A circle with diameter AC intersects the hypotenuse AB at K. If $BK : AK = 1 : 3$, find the measure of the angle $\angle BAC$.

Solution: The triangle AKC is drawn in a semi-circle, so the angle AKC is a right angle. Thus, AKC and ACB are similar triangles. WOLOG we assume that $AK = 3$ and $BK = 1$. Let $AC = x$. Then by similar triangles $\frac{3}{x} = \frac{x}{4}$, so $x = 2\sqrt{3}$. By the Pythagorean Theorem, $BC^2 = 4^2 - 12 = 4$ so $BC = 2$. We see that ABC is a $30 - 60 - 90$ triangle, with BAC being the 30° angle.

3. An arithmetic sequence is a sequence where each term after the first is the sum of the previous term plus a constant value. For example, $3, 7, 11, 15, \ldots$ is an arithmetic sequence.
   S is a sequence which has the following properties:
   - The first term of S is positive.
   - The first three terms of S form an arithmetic sequence.
   - If a square is constructed with area equal to a term in S, then the perimeter of that square is the next term in S.

   Determine all possible values for the third term of S.

Solution: Let the first term be a^4. By the third condition, the second term must be $4a^2$ and the third term must be $8a$. In an arithmetic sequence, the first term added to the third is twice the second term. So in our sequence we get:

$$\begin{aligned} a^4 + 8a &= 8a^2 \\ a^4 - 8a^2 + 8a &= 0 \\ a(a-2)(a^2+2a-4) &= 0 \end{aligned}$$

This gives solutions of $a = 0, 2, -1 \pm \sqrt{5}$. Since each of our terms represents the area or perimeter of a square, we need $a > 0$. Thus, $a = 2$ and $a = \sqrt{5} - 1$, which gives 16 and $8\sqrt{5} - 8$ as the possible values for the third term.

4. A farmer has a flock of n sheep, where $2000 \leq n \leq 2100$. The farmer puts some number of the sheep into one barn and the rest of the sheep into a second barn. The farmer realizes that if she were to select two different sheep at random from her flock, the probability that they are in different barns is exactly $\frac{1}{2}$. Determine the value of n.

Solution 1: Suppose the farmer has n sheep total and puts k sheep into one of the barns. The probability that the farmer selects sheep from different barns is $\frac{k(n-k)}{\binom{n}{2}}$.

$$\begin{aligned}
\frac{k(n-k)}{\binom{n}{2}} &= \frac{1}{2} \\
2k(n-k) &= n(n-1)/2 \\
(4k)n - 4k^2 &= n^2 - n \\
n^2 - (4k+1)n + 4k^2 &= 0 \\
n &= \frac{(4k+1) \pm \sqrt{(4k+1)^2 - 16k^2}}{2} \\
n &= \frac{(4k+1) \pm \sqrt{8k+1}}{2}
\end{aligned}$$

In order for n to be an integer, it is necessary to have $8k + 1 = a^2$ for some positive integer a, so we must have $k = \frac{a^2-1}{8}$. Substituting this back into our expression for n gives:

$$\begin{aligned}
n &= \frac{4a^2 - 4 + 8 \pm 8a}{16} \\
&= \frac{4a^2 + 4 \pm 8a}{16} \\
&= \frac{a^2 \pm 2a + 1}{4} \\
&= \left(\frac{a \pm 1}{2}\right)^2
\end{aligned}$$

Thus, n must be a perfect square. The only perfect square between 2000 and 2100 is 2025, and when $a = 89$ we get $n = 2025, k = 990$ as a solution. Thus, $n = 2025$.

Solution 2: We proceed as in solution 1, getting $n^2 - (4k+1)n + 4k^2 = 0$ which we can rewrite as $n = n^2 - 4k + 4k^2$. This simplifies to $n = (n - 2k)^2$, and so n must be a perfect square, meaning $n = 2025$.

Section C – 10 marks each

1. A quadratic polynomial $f(x) = x^2 + px + q$, with p and q real numbers, is said to be a *double-up polynomial* if it has two real roots, one of which is twice the other.

   (a) If a double-up polynomial $f(x)$ has $p = -15$, determine the value of q.

   (b) If $f(x)$ is a double-up polynomial with one of the roots equal to 4, determine all possible values of $p + q$.

   (c) Determine all double-up polynomials for which $p + q = 9$.

Solution:

(a) Suppose a double-up polynomial has two roots which are k and $2k$. Then the polynomial is $(x - k)(x - 2k) = x^2 - 3k + 2k^2$. When $p = -15$, we have $k = 5$ so $q = 2 \cdot (5)^2 = 50$.

(b) From the previous part, we know that $p + q = 2k^2 - 3k$. If one root is equal to 4 then from above we either have $k = 4$ or $k = 2$. When $k = 4$ we get $p + q = 32 - 12 = 20$ and when $k = 2$ we get $p + q = 8 - 6 = 2$.

(c) When $p + q = 9$ we have $2k^2 - 3k = 9$ which has solutions $k = 3, k = -\frac{3}{2}$. We get $x^2 - 9x + 18$ and $x^2 + \frac{9}{2}x + \frac{9}{2}$ as the polynomials.

2. Let $O = (0, 0), Q = (13, 4), A = (a, a), B = (b, 0)$, where a and b are positive real numbers with $b \geq a$. The point Q is on the line segment AB.

   (a) Determine the values of a and b for which Q is the midpoint of AB.

   (b) Determine all values of a and b for which Q is on the line segment AB and the triangle OAB is isosceles and right-angled.

   (c) There are infinitely many line segments AB that contain the point Q. For how many of these line segments are a and b both integers?

Solution:

(a) For Q to be the midpoint of AB we have $(a + b)/2 = 13$ and $(a + 0)/2 = 4$. The second equation means that $a = 8$, which we can substitute into the first equation to get $b = 18$.

(b) When OAB is isosceles, there are 3 possibilities: $OA = OB, OA = AB, OB = AB$.

When $OB = AB$ we have a right angled triangle with a right angle at b, which would mean $a = b$. For the point Q to be on this line, we must have $a = b = 13$.

When $OA = AB$, we have a right angled triangle at A. By symmetry we have $b = 2a$. So we need the point $(13, 4)$ to be on the line through (a, a) and $(2a, 0)$. This is the line$x + y = 2a$ and since $(13, 4)$ is on the line, we have $a = 8.5, b = 17$.

When $OA = OB$ the triangle is not right-angled.

(c) The slope of the line segment AQ is $\frac{4-a}{13-a}$ and the slope of the line segment QB is $\frac{0-4}{b-13}$. These two line segments have the same slope, so we have that:

$$\begin{aligned}
\tfrac{-4}{b-13} &= \tfrac{4-a}{13-a} \\
52 - 4a &= ab - 4b - 13a + 52 \\
0 &= ab - 9a - 4b \\
36 &= (a - 4)(b - 9)
\end{aligned}$$

For a and b to be positive integers, each of $a - 4$ and $b - 9$ must be positive divisors of 36. There are 9 possibilities:

a - 4	b - 9	a	b
1	36	5	45
2	18	6	27
3	12	7	21
4	9	8	18
6	6	10	15
9	4	13	13
12	3	16	12
18	2	22	11
36	1	40	10

Since $b \geq a$ this excludes the last 3 cases, so there are 6 such line segments AB.

3. (a) If $n = 3$, determine all integer values of m such that $m^2 + n^2 + 1$ is divisible by $m - n + 1$ and $m + n + 1$.

   (b) Show that for any integer n there is always at least one integer value of m for which $m^2 + n^2 + 1$ is divisible by both $m - n + 1$ and $m + n + 1$.

   (c) Show that for any integer n there are only a finite number of integer values m for which $m^2 + n^2 + 1$ is divisible by both $m - n + 1$ and $m + n + 1$.

Solution:

(a) When $n = 3$ we need $m^2 + 10$ divisible by $m - 2$ and $m + 4$. We can write $m^2 + 10 = (m - 2)(m + 2) + 14$, meaning $m - 2$ must divide 14. Similarly, we can write $m^2 + 10 = (m + 4)(m - 4) + 26$, meaning $m + 4$ must divide 26. Since $m - 2$ divides 14, $m - 2$ must be one of $-14, -7, -2, -1, 1, 2, 7, 14$ and m must be one of $-12, -5, 0, 1, 3, 4, 9, 16$. Similarly since $m + 4$ divides 26, m must be one of $-30, -17, -6, -5, -3, -2, 9, 22$. The only values of m that are common to the two lists are $m = -5$ and $m = 9$.

(b) When $m = n^2$, we have

$$m^2 + n^2 + 1 = n^4 + n^2 + 1 = (n^2 + n + 1)(n^2 - n + 1).$$

This is clearly divisible by $m + n + 1$ and $m - n + 1$ for every integer n.

(c) We can write $m^2 + n^2 + 1 = (m - n + 1)(m + n - 1) + 2(n^2 - n + 1)$. Thus, for $m - n + 1$ to divide $m^2 + n^2 + 1$ it must also divide $2(n^2 - n + 1)$. For any given value of n, $2(n^2 - n + 1) > 0$ and so has finitely many positive divisors, hence there are only finitely many possible values of m.

4. Mr. Whitlock is playing a game with his math class to teach them about money. Mr. Whitlock's math class consists of $n \geq 2$ students, whom he has numbered from 1 to n. Mr. Whitlock gives $m_i \geq 0$ dollars to student i, for each $1 \leq i \leq n$, where each m_i is an integer and $m_1 + m_2 + \cdots + m_n \geq 1$.
   We say a student is a *giver* if no other student has more money than they do and we say a student is a *receiver* if no other student has less money than they do. To play the game, each student who is a giver, gives one dollar to each student who is a receiver (it is possible for a student to have a negative amount of money after doing so). This process is repeated until either all students have the same amount of money, or the students reach a distribution of money that they had previously reached.

   (a) Give values of $n, m_1, m_2, \ldots, m_n$ for which the game ends with at least one student having a negative amount of money, and show that the game does indeed end this way.

   (b) Suppose there are n students. Determine the smallest possible value k_n such that if $m_1 + m_2 + \cdots + m_n \geq k_n$ then no player will ever have a negative amount of money.

   (c) Suppose $n = 5$. Determine all quintuples $(m_1, m_2, m_3, m_4, m_5)$, with $m_1 \leq m_2 \leq m_3 \leq m_4 \leq m_5$, for which the game ends with all students having the same amount of money.

Solution:

(a) Suppose there are 5 students and the starting amounts are: $(0, 0, 0, 1, 2)$. As the game progresses, the amounts of money become: $(1, 1, 1, 1, -1)$ then $(0, 0, 0, 0, 3)$ and finally $(1, 1, 1, 1, -1)$ at which point the game ends.

(b) Consider a game with n players that starts with the initial distribution of one player having $n - 2$ dollars and all the rest having $n - 3$. After the first turn, the first player will have -1 dollars, thus $n - 2 + (n - 3)(n - 1) = n^2 - 3n + 1$ is not sufficient.

We observe that the only way for a person to end up with negative money is if they were the giver and gave away more money than they had. Suppose instead that the sum of all the money is $n^2 - 3n + 2$. By the pigeonhole principle, at every stage of the game, we must either have at least one student with at least $n - 1$ dollars or at least two students with exactly $n - 2$ dollars. In the first case, since the giver is giving one dollar to at most $n - 1$ people, they will not end the turn with a negative amount of money. Similarly, if there are at least two students with $n - 2$ dollars, then they will each give one dollar to at most $n - 2$ students and will not end the turn with a negative amount of money. Thus $k_n = n^2 - 3n + 2$.

(c) We observe that it is clearly necessary for $m_1 + m_2 + m_3 + m_4 + m_5$ to be a multiple of 5 in order for the students to all have the same amount of money, so we will assume that this is the case. Let m be the average amount of money. We see that if a student ever has exactly m dollars, they will have m dollars for the remainder of the game, since either everyone will have m dollars or there will be someone with more money and someone with less money.

We first consider the following scenarios:

- $m_1 = m_2 = m_3 = m_4 < m_5$ As long as $m_5 \neq m_1$ four player will gain a dollar while the 5th player loses 4 dollars. This will continue until the first four players have each have m at which point the game will end with all players having the same amount of money. The $m_2 = m_3 = m_4 = m_5$ case is symmetric.
- $m_1 < m_2 = m_3 = m_4 < m_5$ Each turn, player 5 will give one dollar to player 1 until one of them as the same amount of money as players 2 through 4, so this game will always end with all players having the same amount of money.
- $m_1 = m_2 = m_3 < m_4 \leq m_5$ If at any point players 1 through 3 are are neither givers nor receivers, then the game will end with all players having the same amount of money, since we will be in the previous case. Since there are only two other players, players 1 through 3 can only gain or lose at most two dollars on a turn. In order for them to always be a giver or receiver, they must never have exactly m dollars, so eventually they must have $m - 1$ dollars each and then alternate between $m - 1$ and $m + 1$. However, if all three have $m - 1$ dollars and are still receivers, then the other players do not have the same amount of money as each other, so they each only gain 1 dollar. Thus, this game must end with all players hvaing the same amount of money.
- $m_1 = m_2 < m_3 < m_4 = m_5$ If $k = \min(m_3 - m_2, m_4 - m_3)$ is even, then after $k/2$ turns there will be three players with the same amount of money, and the game will end with all players having the same amount of money.
  If k is odd then after $(k - 1)/2$ turns without loss of generality the players will have $n_1 = n_2 < n_3 < n_4 = n_5$ dollars respectively, with $n_3 = n_2 + 1$. Since the total amount of money is divisible by 5 we get that $n_4 = n_5 = n_2 + 2p + 2$ for some non-negative integer p. If $p > 0$ then the dollar amounts after the next turn is $(n_2+2, n_2+2, n_2+1, n_2+2p, n_2+2p)$ and then $(n_2+2, n_2+2, n_2+3, n_2+2(p-1)+4, n_2+2(p-1)+4)$ which is the same as $(q_2, q_2, q_2+1, q_2+2(p-1)+2, q_2+2(p-1)+2)$. Thus, eventually we will get to a point where the players have $(m - 1, m - 1, m, m + 1, m + 1)$ dollars respectively, and this game will not end with all players having the same amount of money.
- $m_1 = m_2 < m_3 = m_4 < m_5$ If $k = m_5 - m_4$ is even then after $k/2$ turns the last three players will all have the same amount of money, unless on a previous turn the first four players all had the same amount of money. In either case, the game ends with all players having the same amount of money.
  If k is odd, then after $(k + 1)/2$ turns we will be at the odd case of the previous scenario, in which case the game ends without all players having the same amount of money, unless $2(m_3 - m_2) < k$, in which case we will be in the first scenario and the game will end with all players having the same amount of money.

We now seek to reduce all cases to one of the above scenarios. Let us assume WOLOG that $m_2 - m_1 \leq m_5 - m_4$. This means that on the first $m_2 - m_1$ turns, player 5 will have given $m_2 - m_1$ dollars to player 1. So after this point the respective dollar amount for the players will be m_2, m_2, m_3, m_4, m_5', where $m_5' = m_5 - m_2 + m_1$.

If $m_5' - m_4 \geq 2(m_3 - m_2)$ then after the next $m_3 = m_2$ turns, the first three players will have the same amount of money so the game ends with all players having the same amount of money.

If $k = m_5' - m_4$ is even then after the next $k/2$ moves players 4 and 5 will both have m_4 dollars and the first two players will have $(m_1 + m_2 + m_5 - m_4)/2$ dollars. This will end with all players having the same amount of money if and only if $\min(m_4 - m_3, m_3 - (m_1 + m_2 + m_5 - m_4)/2$ is even. Note that k is even if and only if $m_1 + m_2 + m_4 + m_5$ is even.

If k is odd, then player 5 will give money to players 1 and 2 until they have only 1 dollar less than player 4. Then player 4 and 5 will take turns giving money to players 1 and 2 until players 1 and 2 have the same amount of money as player 3, or one of players 4 and 5 has the same amount of money as player 3. If player 1 and 2 have the same amount as player 3, the game will end with all players having the same amount of money. This occurs when $m_5 + m_4 - 2m_3 > 2m_3 - m_2 - m_1$. If either 4 or 5 has the same amount has 3, the game will not end with all the players having the same amount if money.

Thus, we see the game will end with all players having the same amount of money when $m_2 - m_1 \leq m_5 - m_4$ and one of the following 3 conditions hold

- $m_5 - m_4 \geq 2m_3 - m_2 - m_1$
- $m_1 + m_2 + m_4 + m_5$ and $\min(m_4 - m_3, m_3 - (m_1 + m_2 + m_5 - m_4)/2)$ are both even
- $m_1 + m_2 + m_4 + m_5$ is odd and $m_5 + m_4 - 2m_3 > 2m_3 - m_2 - m_1$

or when $m_2 - m_1 > m_5 - m_4$ and one of the following 3 conditions hold

- $m_2 - m_1 \geq m_5 + m_4 - 2m_3$
- $m_1 + m_2 + m_4 + m_5$ and $\min(m_3 - m_2, (m_1 - m_2 + m_4 + m_5)/2 - m_3)$ are both even
- $m_1 + m_2 + m_4 + m_5$ is odd and $2m_3 - m_2 - m_1 > m_5 + m_4 - 2m_1$.

