CMS SMC

Sun Life Financial

The 2012 Sun Life Financial Canadian Open Mathematics Challenge

Please print clearly and complete all information below. Failure to print legibly or provide complete information may result in your exam being disqualified. This exam is not considered valid unless it is accompanied by your test supervisor's signed form.

First Name:

Last Name:

Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since September 15th of this year?

☐ Y ☐ N

Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of current address)?

☐ Y ☐ N

Grade

☐ 8 ☐ 9 ☐ 10
☐ 11 ☐ 12 ☐ Cégep

Other: ________

T-Shirt Size (Youth)

☐ XS ☐ S ☐ M
☐ L ☐ XL ☐ XXL

Date of Birth: y y y y m m d d

Gender: (Optional)

☐ M ☐ F

E-mail Address:

Signature: ________________________________

INSTRUCTIONS: DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO

Mobile phones and calculators are not permitted.

EXAM: There are 3 parts to the COMC to be completed in 2 hours and 30 minutes.
PART A: Consists of 4 basic questions worth 4 marks each.
PART B: Consists of 4 intermediate questions worth 6 marks each.
PART C: Consists of 4 advanced questions worth 10 marks each.

DIAGRAMS: Diagrams are not drawn to scale; they are intended as aids only.

WORK AND ANSWERS: All solution work and answers are to be presented in this booklet in the boxes provided. Marks are awarded for completeness and clarity. For sections A and B, it is not necessary to show your work in order to receive full marks. However, if your answer or solution is incorrect, any work that you do and present in this booklet will be considered for part marks. For section C, you must show your work and provide the correct answer or solution to receive full marks.

It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2 + \sqrt{7}$, etc., rather than as 12.566, 4.646, etc. The names of all award winners will be published on the Canadian Mathematical Society web site.

The contents of the COMC 2012 and your answers and solutions must not be publically discussed, including web chats, for at least 24 hours.

For official use only.

A1	A2	A3	A4	A	B1	B2	B3	B4	B	C1	C2	C3	C4	C	ABC

______ Marker initials ______ Data entry initials

Part A: Question 1 (4 marks)

Determine the positive integer n such that $8^4 = 4^n$.

Your Solution:

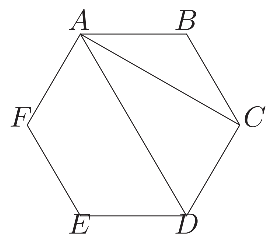
Part A: Question 2 (4 marks)

Let x be the *average* of the following six numbers: $\{12, 412, 812, 1212, 1612, 2012\}$. Determine the value of x.

Your Solution:

Part A: Question 3 (4 marks)

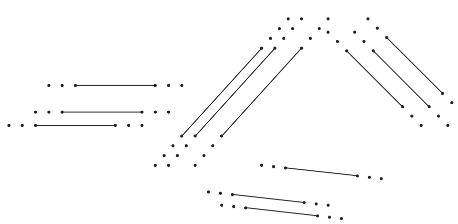
Let $ABCDEF$ be a hexagon all of whose sides are equal in length and all of whose angles are equal. The area of hexagon $ABCDEF$ is exactly r times the area of triangle ACD. Determine the value of r.



Your Solution:

Part A: Question 4 (4 marks)

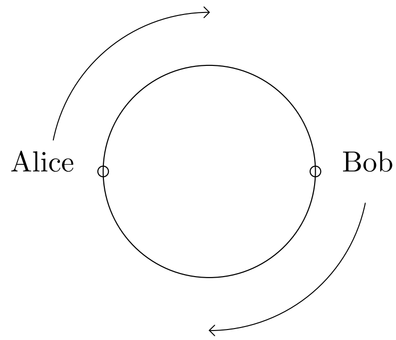
Twelve different lines are drawn on the coordinate plane so that each line is parallel to exactly two other lines. Furthermore, no three lines intersect at a point. Determine the total number of intersection points among the twelve lines.



Your Solution:

Part B: Question 1 (6 marks)

Alice and Bob run in the clockwise direction around a circular track, each running at a constant speed. Alice can complete a lap in t seconds, and Bob can complete a lap in 60 seconds. They start at diametrically-opposite points.



When they meet for the first time, Alice has completed exactly 30 laps. Determine all possible values of t.

Your Solution:

Part B: Question 2 (6 marks)

For each positive integer n, define $\varphi(n)$ to be the number of positive divisors of n. For example, $\varphi(10) = 4$, since 10 has 4 positive divisors, namely $\{1, 2, 5, 10\}$.

Suppose n is a positive integer such that $\varphi(2n) = 6$. Determine the minimum possible value of $\varphi(6n)$.

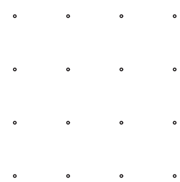
Your Solution:

Part B: Question 3 (6 marks)

Given the following 4 by 4 square grid of points, determine the number of ways we can label ten different points $A, B, C, D, E, F, G, H, I, J$ such that the lengths of the nine segments

$$AB, BC, CD, DE, EF, FG, GH, HI, IJ$$

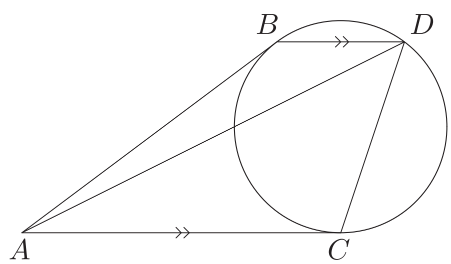
are in strictly increasing order.



Your Solution:

Part B: Question 4 (6 marks)

In the following diagram, two lines that meet at a point A are tangent to a circle at points B and C. The line parallel to AC passing through B meets the circle again at D. Join the segments CD and AD. Suppose $AB = 49$ and $CD = 28$. The length of AD is a positive integer n. Determine n.



Your Solution:

Part C: Question 1 (10 marks)

Let $f(x) = x^2$ and $g(x) = 3x - 8$.

(a) (2 marks) Determine the values of $f(2)$ and $g(f(2))$.

(b) (4 marks) Determine all values of x such that $f(g(x)) = g(f(x))$.

(c) (4 marks) Let $h(x) = 3x - r$. Determine all values of r such that $f(h(2)) = h(f(2))$.

Your Solution:

Part C: Question 2 (10 marks)

C2 We fill a 3×3 grid with 0s and 1s. We score one point for each row, column, and diagonal whose sum is *odd*.

1	1	0
1	0	1
0	1	1

1	1	1
1	0	1
0	1	1

For example, the grid on the left has 0 points and the grid on the right has 3 points.

(a) (2 marks) Fill in the following grid so that the grid has exactly 1 point. No additional work is required. Many answers are possible. You only need to provide one.

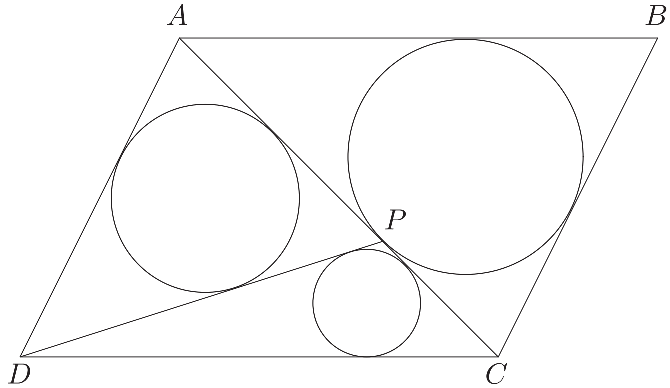
(b) (4 marks) Determine all grids with exactly 8 points.

(c) (4 marks) Let E be the number of grids with an even number of points, and O be the number of grids with an odd number of points. Prove that $E = O$.

Your Solution:

Part C: Question 3 (10 marks)

Let $ABCD$ be a parallelogram. We draw in the diagonal AC. A circle is drawn inside ΔABC tangent to all three sides and touches side AC at a point P.



(a) (2 marks) Prove that $DA + AP = DC + CP$.

(b) (4 marks) Draw in the line DP. A circle of radius r_1 is drawn inside ΔDAP tangent to all three sides. A circle of radius r_2 is drawn inside ΔDCP tangent to all three sides. Prove that

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

(c) (4 marks) Suppose $DA + DC = 3AC$ and $DA = DP$. Let r_1, r_2 be the two radii defined in (b). Determine the ratio r_1/r_2.

Your Solution:

Part C: Question 4 (10 marks)

For any positive integer n, an n-tuple of positive integers $(x_1, x_2, \cdots, x_n)$ is said to be *super-squared* if it satisfies both of the following properties:

(1) $x_1 > x_2 > x_3 > \cdots > x_n$.

(2) The sum $x_1^2 + x_2^2 + \cdots + x_k^2$ is a perfect square for each $1 \leq k \leq n$.

For example, $(12, 9, 8)$ is super-squared, since $12 > 9 > 8$, and each of 12^2, $12^2 + 9^2$, and $12^2 + 9^2 + 8^2$ are perfect squares.

(a) (2 marks) Determine all values of t such that $(32, t, 9)$ is super-squared.

(b) (2 marks) Determine a super-squared 4-tuple (x_1, x_2, x_3, x_4) with $x_1 < 200$.

(c) (6 marks) Determine whether there exists a super-squared 2012-tuple.

Your Solution:

CMS SMC Canadian Mathematical Society
Société mathématique du Canada

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Canadian Open Mathematics Challenge

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Carleton University

Dalhousie University
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Université Laval

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