The Canadian Mathematical Society | La Société mathématique du Canada

The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

Sun Life Financial Canadian Open Mathematics Challenge

Wednesday, November 24, 2010

Solutions

Part A

1. *Solution 1*

   Calculating, $\dfrac{(9+5)^2-(9-5)^2}{(9)(5)} = \dfrac{14^2-4^2}{45} = \dfrac{196-16}{45} = \dfrac{180}{45} = 4.$

   Solution 2

   For a general x and y with x and y not zero,

   $$\frac{(x+y)^2-(x-y)^2}{xy} = \frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{xy} = \frac{4xy}{xy} = 4$$

   Since this expression equals 4 for any values of x and y, then $\dfrac{(9+5)^2-(9-5)^2}{(9)(5)} = 4.$

   Solution 3

   For a general x and y with x and y not zero, we can factor as a difference of squares:

   $$\frac{(x+y)^2-(x-y)^2}{xy} = \frac{[(x+y)+(x-y)][(x+y)-(x-y)]}{xy} = \frac{(2x)(2y)}{xy} = 4$$

   Since this expression equals 4 for any values of x and y, then $\dfrac{(9+5)^2-(9-5)^2}{(9)(5)} = 4.$

   Answer: 4

2. Simplifying both sides,

   $$\begin{aligned} x-(8-x) &= 8-(x-8) \\ x-8+x &= 8-x+8 \\ 3x &= 24 \\ x &= 8 \end{aligned}$$

   Therefore, $x = 8$.

   Answer: $x = 8$

3. *Solution 1*

   We call the ring between the middle and inner circles the "inner ring".
   We reflect the shaded portion of the inner ring across line segment CD. The area of the shaded region does not change when we do this.
   The shaded region is now the entire semi-circle to the right of CD.
   Thus, the area of the shaded region is half of the area of the outer circle.
   Since $OC = 6$, then the outer circle has radius 6 and so has area $\pi 6^2 = 36\pi$.
   Therefore, the area of the shaded region is $\frac{1}{2}(36\pi) = 18\pi$.

Solution 2

We call the ring between the outer and middle circles the "outer ring", and the ring between the middle and inner circles the "inner ring".

Since $OC = 6$, then the outer circle has radius 6 and so has area $\pi 6^2 = 36\pi$.

Since $OB = 4$, then the middle circle has radius 4 and so has area $\pi 4^2 = 16\pi$.

Since $OA = 2$, then the inner circle has radius 2 and so has area $\pi 2^2 = 4\pi$.

Since the outer circle has area 36π and the middle circle has area 16π, then the area of the outer ring is $36\pi - 16\pi = 20\pi$.

Since the diameter CD divides each ring into two parts of equal area, then the shaded region of the outer ring has area $\frac{1}{2}(20\pi) = 10\pi$.

Since the middle circle has area 16π and the inner circle has area 4π, then the area of the inner ring is $16\pi - 4\pi = 12\pi$.

Since the diameter CD divides each ring into two parts of equal area, then the shaded region of the inner ring has area $\frac{1}{2}(12\pi) = 6\pi$.

Since the inner circle has area 4π and line segment CD passes through the centre of this circle, then the shaded region of the inner circle has area $\frac{1}{2}(4\pi) = 2\pi$.

Therefore, the total shaded area is $10\pi + 6\pi + 2\pi = 18\pi$.

Answer: 18π

4. First, we simplify the given expression:

$$\frac{(3.1 \times 10^7)(8 \times 10^8)}{2 \times 10^3} = \frac{3.1 \times 8}{2} \times \frac{10^7 \times 10^8}{10^3} = 3.1 \times 4 \times 10^{7+8-3} = 12.4 \times 10^{12} = 124 \times 10^{11}$$

Therefore, this integer consists of the digits 124 followed by 11 zeroes, so has 14 digits.

Answer: 14

5. *Solution 1*

Let Q be the point on the line $y = x$ that is closest to $P(-3, 9)$.

Then PQ is perpendicular to the line $y = x$.

Since the line with equation $y = x$ has slope 1 and PQ is perpendicular to this line, then PQ has slope -1.

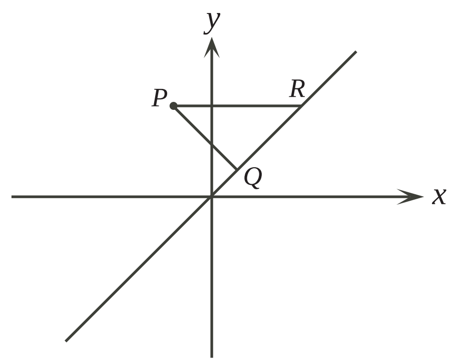
Note that a general point Q on the line with equation $y = x$ has coordinates (t, t) for some real number t.

For the slope of PQ to equal -1, we must have $\dfrac{t-9}{t-(-3)} = -1$ or $t - 9 = -(t+3)$ or $2t = 6$ or $t = 3$.

Therefore, the point on the line with equation $y = x$ that is closest to P is the point $(3, 3)$.

Solution 2

Let Q be the point on the line $y = x$ that is closest to $P(-3, 9)$.
Then PQ is perpendicular to the line $y = x$.
Let R be the point on the line $y = x$ so that PR is horizontal, as shown.



Since PR is horizontal and P has y-coordinate 9, then R has y-coordinate 9.
Since R lies on the line with equation $y = x$, then R has coordinates $(9, 9)$.
Since PR is horizontal and QR has slope 1 (because it lies along the line with equation $y = x$), then $\angle PRQ = 45^\circ$.
Since $\angle PQR = 90^\circ$, then $\triangle PQR$ is isosceles and right-angled.
Let M be the midpoint of PR. Since P has coordinates $(-3, 9)$ and R has coordinates $(9, 9)$, then M has coordinates $(3, 9)$.
Since $\triangle PQR$ is isosceles and M is the midpoint of PR, then QM is perpendicular to PR.
Thus, QM is vertical, so Q has x-coordinate 3.
Since Q lies on the line with equation $y = x$, then Q has coordinates $(3, 3)$.

Solution 3

Note that a general point Q on the line with equation $y = x$ has coordinates (t, t) for some real number t.
Then $PQ = \sqrt{(t - (-3))^2 + (t - 9)^2}$ or $PQ^2 = (t + 3)^2 + (t - 9)^2$.
Since we want to find the point on the line with equation $y = x$ that is closest to P, then we want to minimize the value of PQ, or equivalently to minimize the value of PQ^2.
In other words, we want to find the value of t that minimizes the value of

$$PQ^2 = t^2 + 6t + 9 + t^2 - 18t + 81 = 2t^2 - 12t + 90$$

Since this equation represents a parabola opening upwards, then its minimum occurs at its vertex, which occurs at $t = -\frac{-12}{2(2)} = 3$. Thus, $t = 3$ minimizes the length of PQ.
Therefore, the point on the line with equation $y = x$ that is closest to P is the point $(3, 3)$.

ANSWER: $(3, 3)$

6. Let x be the number of people who studied for the exam and let y be the number of people who did not study.

   We assume without loss of generality that the exam was out of 100 marks.

   Since the average of those who studied was 90%, then those who studied obtained a total of $90x$ marks.

   Since the average of those who did not study was 40%, then those who did not study obtained a total of $40y$ marks.

   Since the overall average was 85%, then $\frac{90x + 40y}{x + y} = 85$.

   Therefore, $90x + 40y = 85x + 85y$ or $5x = 45y$ or $x = 9y$.

   Therefore, $x : y = 9 : 1 = 90 : 10$. This means that 10% of the class did not study for the exam.

   Answer: 10%

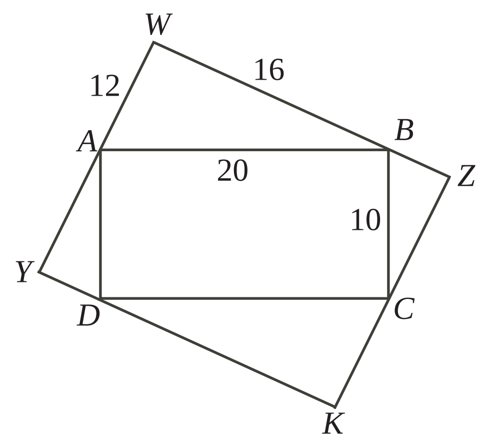
7. *Solution 1*

   Since $ABCD$ is a rectangle, then $AD = BC = 10$ and $DC = AB = 20$.

   Since $WA = 12$, $WB = 16$, $AB = 20$, and $12^2 + 16^2 = 144 + 256 = 400 = 20^2$, then $WA^2 + WB^2 = AB^2$. Thus, $\triangle AWB$ is right-angled at W.

   Note that $\triangle CKD$ is congruent to $\triangle AWB$, so $\triangle CKD$ is right-angled at K.

   Extend WA and KD to meet at Y and WB and KC to meet at Z.



   Suppose that $\angle WAB = \angle KCD = \theta$. Then $\angle WBA = \angle KDC = 90^\circ - \theta$.

   Now $\angle YAD = 180^\circ - \angle WAB - \angle BAD = 180^\circ - \theta - 90^\circ = 90^\circ - \theta$.

   Also, $\angle YDA = 180^\circ - \angle KDC - \angle ADC = 180^\circ - (90^\circ - \theta) - 90^\circ = \theta$.

   Therefore, $\triangle YDA$ is similar to $\triangle WAB$. This means that $\angle DYA = 90^\circ$.

   Also, since $DA = \frac{1}{2}AB$, then the sides of $\triangle YDA$ are half as long as the corresponding sides of $\triangle WAB$. Thus, $YD = \frac{1}{2}WA = 6$ and $YA = \frac{1}{2}WB = 8$.

   Similarly, $\angle BZC = 90^\circ$. Therefore, $WYKZ$ is a rectangle.

   We have $WY = WA + AY = 12 + 8 = 20$ and $YK = YD + DK = 6 + 16 = 22$.

By the Pythagorean Theorem, since $WK > 0$, then

$$WK = \sqrt{WY^2 + YK^2} = \sqrt{20^2 + 22^2} = \sqrt{400 + 484} = \sqrt{884} = 2\sqrt{221}$$

as required.

Solution 2

Since $ABCD$ is a rectangle, then $AD = BC = 10$ and $DC = AB = 20$.

We coordinatize the diagram, putting D at the origin, A at $(0, 10)$, C at $(20, 0)$, and B at $(20, 10)$.

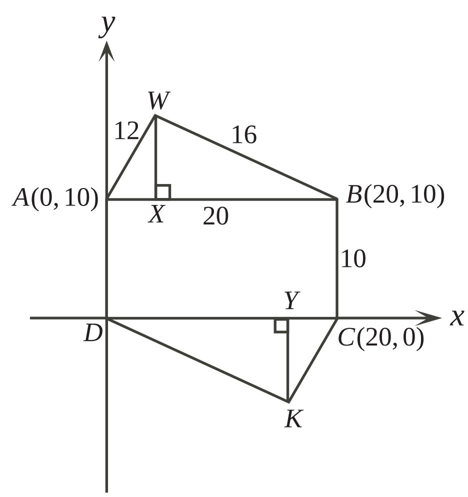
Since $WA = 12$, $WB = 16$, $AB = 20$, and $12^2 + 16^2 = 144 + 256 = 400 = 20^2$, then $WA^2 + WB^2 = AB^2$. Thus, $\triangle AWB$ is right-angled at W.

Note that $\triangle CKD$ is congruent to $\triangle AWB$, so $\triangle CKD$ is right-angled at K.

Since $\triangle AWB$ is right-angled and we know its side lengths, then we can compute the trigonometric ratios of its angles.

In particular, $\sin(\angle WAB) = \dfrac{WB}{AB} = \dfrac{16}{20} = \dfrac{4}{5}$ and $\cos(\angle WAB) = \dfrac{WA}{AB} = \dfrac{12}{20} = \dfrac{3}{5}$.

We drop perpendiculars from W to X on AB and from K to Y on DC.



Then $AX = WA\cos(\angle WAB) = 12(\frac{3}{5}) = \frac{36}{5}$ and $WX = WA\sin(\angle WAB) = 12(\frac{4}{5}) = \frac{48}{5}$.

Since A has coordinates $(0, 10)$, then W has coordinates $(\frac{36}{5}, 10 + \frac{48}{5}) = (\frac{36}{5}, \frac{98}{5})$.

Since $\triangle CKD$ is congruent to $\triangle AWB$, then in a similar way we can find that the coordinates of K are $(20 - \frac{36}{5}, -\frac{48}{5}) = (\frac{64}{5}, -\frac{48}{5})$.

Since we have the coordinates of W and K, then the distance between W and K is

$$\begin{aligned} WK &= \sqrt{(\tfrac{64}{5} - \tfrac{36}{5})^2 + (-\tfrac{48}{5} - \tfrac{98}{5})^2} = \sqrt{\tfrac{28^2}{5^2} + \tfrac{146^2}{5^2}} \\ &= \tfrac{2}{5}\sqrt{14^2 + 73^2} = \tfrac{2}{5}\sqrt{196 + 5329} \\ &= \tfrac{2}{5}\sqrt{5525} = 2\sqrt{\tfrac{5525}{25}} \\ &= 2\sqrt{221} \end{aligned}$$

Therefore, $WK = 2\sqrt{221}$.

ANSWER: $WK = 2\sqrt{221}$

8. *Solution 1*

First, we factor the first and third quadratic factors to obtain

$$(x+1)(x+2)(x^2-2x-1)(x-3)(x-4)+24=0$$

Next, we rearrange the factors to obtain

$$(x+1)(x-3)(x^2-2x-1)(x+2)(x-4)+24=0$$

and expand to obtain

$$(x^2-2x-3)(x^2-2x-1)(x^2-2x-8)+24=0$$

Next, we make the substitution $w = x^2 - 2x$ to obtain

$$(w-3)(w-1)(w-8)+24=0$$

This is a cubic equation in w so we expand, simplify and factor:

$$\begin{aligned} (w^2-4w+3)(w-8)+24 &= 0 \\ w^3-12w^2+35w &= 0 \\ w(w^2-12w+35) &= 0 \\ w(w-5)(w-7) &= 0 \end{aligned}$$

Therefore, the solutions in terms of w are $w = 0$ or $w = 5$ or $w = 7$.
If $w = x^2 - 2x = 0$, then $x(x-2) = 0$ which gives $x = 0$ or $x = 2$.
If $w = x^2 - 2x = 5$, then $x^2 - 2x - 5 = 0$.
The quadratic formula gives the roots $x = \dfrac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$.
If $w = x^2 - 2x = 7$, then $x^2 - 2x - 7 = 0$.
The quadratic formula gives the roots $x = \dfrac{2 \pm \sqrt{32}}{2} = 1 \pm \sqrt{8}$.
Therefore, $x = 0$ or $x = 2$ or $x = 1 \pm \sqrt{6}$ or $x = 1 \pm \sqrt{8}$. (This last pair can be rewritten as $x = 1 \pm 2\sqrt{2}$.)

Solution 2

First, we factor the first and third quadratic factors and complete the square in the second quadratic factor to obtain

$$(x+1)(x+2)((x-1)^2-2)(x-3)(x-4)+24=0$$

Next, we make the substitution $y = x - 1$ (which makes $x + 1 = y + 2$ and $x + 2 = y + 3$ and $x - 3 = y - 2$ and $x - 4 = y - 3$) to obtain

$$(y + 2)(y + 3)(y^2 - 2)(y - 2)(y - 3) + 24 = 0$$

(We made this substitution because it made the algebra more "symmetric"; that is, after making this substitution, factors of the form $y - a$ are paired with factors of the form $y + a$.)
Next, we rearrange the factors to obtain

$$(y + 2)(y - 2)(y + 3)(y - 3)(y^2 - 2) + 24 = 0$$

Next, we multiply out pairs of factors to obtain

$$(y^2 - 4)(y^2 - 9)(y^2 - 2) + 24 = 0$$

Next, we make the substitution $z = y^2$ to obtain

$$(z - 4)(z - 9)(z - 2) + 24 = 0$$

This is a cubic equation in z so we expand and simplify:

$$\begin{aligned} (z^2 - 13z + 36)(z - 2) + 24 &= 0 \\ z^3 - 15z^2 + 62z - 48 &= 0 \end{aligned}$$

By inspection, $z = 1$ is a solution so $z - 1$ is a factor of the cubic equation.
We factor out this linear factor to obtain $(z - 1)(z^2 - 14z + 48) = 0$.
The quadratic factor can be factored as $(z - 6)(z - 8)$.
Therefore, we have $(z - 1)(z - 6)(z - 8) = 0$.
Therefore, the solutions in terms of z are $z = 1$ or $z = 6$ or $z = 8$.
Since $z = y^2$, then the solutions in terms of y are $y = \pm 1$ or $y = \pm\sqrt{6}$ or $y = \pm\sqrt{8}$.
Since $y = x - 1$, then $x = y + 1$, and so the solutions in terms of x are $x = 0$ or $x = 2$ or $x = 1 \pm \sqrt{6}$ or $x = 1 \pm \sqrt{8}$. (This last pair can be rewritten as $x = 1 \pm 2\sqrt{2}$.)

ANSWER: $x = 0, 2, 1 \pm \sqrt{6}, 1 \pm 2\sqrt{2}$

Part B

1. (a) *Solution 1*

From the first row, $A + A = 50$ or $A = 25$.

From the second column, $A + C = 57$. Since $A = 25$, then $C = 57 - 25 = 32$.

Solution 2

From the first row, $A + A = 50$ or $A = 25$.

From the first column, $A + B = 37$. Since $A = 25$, then $B = 37 - 25 = 12$.

From the second row, $B + C = 44$. Since $B = 12$, then $C = 44 - 12 = 32$.

(b) *Solution 1*

The sum of the nine entries in the table equals the sum of the column sums, or $50+n+40 = 90 + n$. (This is because each entry in the table is part of exactly one column sum.)

Similarly, the sum of the nine entries in the table also equals the sum of the row sums, or $30 + 55 + 50 = 135$.

Therefore, $90 + n = 135$ or $n = 45$.

Solution 2

The sum of the nine entries in the table equals the sum of the row sums, or $30+55+50 = 135$. (This is because each entry in the table is part of exactly one row sum.)

Since the entries in the table include three entries equal to each of D, E and F, then the sum of the entries in the table is also $3D + 3E + 3F = 3(D + E + F)$.

Therefore, $3(D + E + F) = 135$ or $D + E + F = 45$.

From the second column, $D + E + F = n$. Thus, $n = 45$.

Solution 3

From the first row, $D + D + D = 30$ or $D = 10$.

From the first column, $D + 2F = 50$. Since $D = 10$, then $2F = 50 - 10$ and so $F = 20$.

From the third column, $D + 2E = 40$. Since $D = 10$, then $2E = 40 - 10$ and so $E = 15$.

Therefore, $n = D + E + F = 10 + 15 + 20 = 45$.

(c) *Solution 1*

From the third row, $3R + T = 33$.

From the fourth row, $R + 3T = 19$.

Adding these equations, we obtain $4R + 4T = 52$ or $R + T = 13$.

From the first row, $P + Q + R + T = 20$.

Since $R + T = 13$, then $P + Q = 20 - 13 = 7$.

Solution 2

The sum of the sixteen entries in the table equals the sum of the row sums, or $20 + 20 + 33 + 19 = 92$. (This is because each entry in the table is part of exactly one row sum.)

The table includes two entries equal to each of P and Q and six entries equal to each of R and T.

Therefore, $2P + 2Q + 6R + 6T = 92$.

The last two rows of the table include four entries equal to each of R and T, so $4R+4T = 33 + 19 = 52$, or $R + T = 13$.

Therefore, $2P + 2Q = 92 - 6(R + T) = 92 - 6(13) = 14$, and so $P + Q = 7$.

Solution 3

From the third row, $3R + T = 33$.

From the fourth row, $R + 3T = 19$.

Multiplying the first equation by 3 and subtracting the second equation gives $(9R + 3T) - (R + 3T) = 99 - 19$ or $8R = 80$ or $R = 10$.

Since $3R + T = 33$, then $T = 33 - 3(10) = 3$.

From the first row, $P + Q + R + T = 20$.

Since $R = 10$ and $T = 3$, then $P + Q = 20 - 10 - 3 = 7$.

2. (a) To determine the coordinates of A and B, we equate values of y using the equations $y = x^2 - 4x + 12$ and $y = -2x + 20$ to obtain

$$\begin{aligned} x^2 - 4x + 12 &= -2x + 20 \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \end{aligned}$$

Therefore, $x = 4$ or $x = -2$.

To determine the y-coordinates of points A and B, we can use the equation of the line.

If $x = 4$, then $y = -2(4) + 20 = 12$.

If $x = -2$, then $y = -2(-2) + 20 = 24$.

Therefore, the coordinates of A and B are $(4, 12)$ and $(-2, 24)$.

(b) Using the coordinates of A and B from (a), the coordinates of the midpoint M of AB are $(\frac{1}{2}(4 + (-2)), \frac{1}{2}(24 + 12))$ or $(1, 18)$.

(c) *Solution 1*

The line with equation $y = -2x + 20$ has slope -2.

Therefore, we have a line with slope -2 that intersects the parabola at points

$P(p, p^2 - 4p + 12)$ and $Q(q, q^2 - 4q + 12)$.
In other words, line segment PQ has slope -2.
Therefore,

$$\begin{aligned}
\frac{(p^2 - 4p + 12) - (q^2 - 4q + 12)}{p - q} &= -2 \\
\frac{p^2 - q^2 - 4p + 4q}{p - q} &= -2 \\
\frac{(p - q)(p + q) - 4(p - q)}{p - q} &= -2 \\
(p + q) - 4 &= -2 \qquad (\text{since } p \neq q) \\
p + q &= 2
\end{aligned}$$

Therefore, $p + q = 2$, as required.

Solution 2
The line with equation $y = -2x + 20$ has slope -2.
Therefore, we have a line with slope -2 (say with equation $y = -2x + b$) that intersects the parabola at points P and Q.
Since $y = -2x + b$ and $y = x^2 - 4x + 12$ intersect when $x = p$, then $p^2 - 4p + 12 = -2p + b$, which gives $p^2 - 2p + 12 - b = 0$.
Since $y = -2x + b$ and $y = x^2 - 4x + 12$ intersect when $x = q$, then $q^2 - 4q + 12 = -2q + b$, which gives $q^2 - 2q + 12 - b = 0$.
Since we have two expressions equal to 0, then

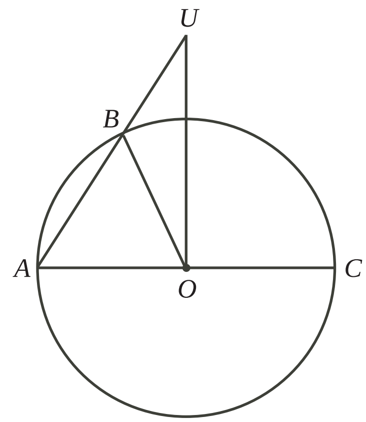
$$\begin{aligned}
p^2 - 2p + 12 - b &= q^2 - 2q + 12 - b \\
p^2 - 2p &= q^2 - 2q \\
p^2 - q^2 - 2p + 2q &= 0 \\
(p - q)(p + q) - 2(p - q) &= 0 \\
(p - q)(p + q - 2) &= 0
\end{aligned}$$

Therefore, $p - q = 0$ or $p + q - 2 = 0$.
Since $p \neq q$, then $p + q = 2$.

(d) Since P has coordinates $(p, p^2 - 4p + 12)$ and Q has coordinates $(q, q^2 - 4q + 12)$, then the x-coordinate of the midpoint N of PQ is $\frac{1}{2}(p + q)$.
Since $p + q = 2$ by (c), then the x-coordinate of N is 1.
Since the x-coordinate of M is 1 and the x-coordinate of N is 1, then line segment MN is vertical.

3. (a) *Solution 1*

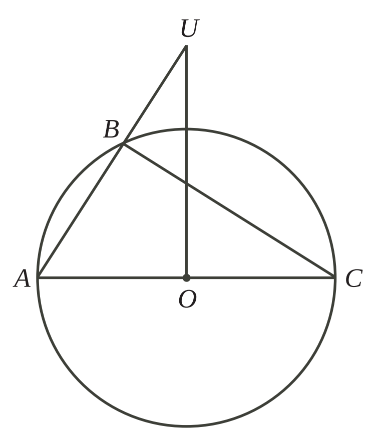
Let U be the point in $\mathcal{S}$ vertically above O and let B be the point where AU intersects the circle. (There will be one other point U in $\mathcal{S}$ with UO perpendicular to AC; this point will be vertically below O. By symmetry, the length of UO is the same in either case.)
Join UO and BO.



Let $\angle BUO = \theta$.
Note that $AO = BO = 1$ since they are radii and $BU = 1$ by definition.
Therefore, $\triangle UBO$ is isosceles and so $\angle BOU = \angle BUO = \theta$.
Now $\angle ABO$ is an exterior angle in this triangle, so $\angle ABO = \angle BUO + \angle BOU = 2\theta$.
Since $OB = OA$, then $\triangle ABO$ is isosceles and so $\angle BAO = \angle ABO = 2\theta$.
But $\triangle UAO$ is right-angled at O, and so $\angle UAO + \angle AUO = 90^\circ$ or $2\theta + \theta = 90^\circ$.
Therefore, $3\theta = 90^\circ$ or $\theta = 30^\circ$.
This tells us that $\triangle UAO$ is a 30°-60°-90° triangle, and so $UO = \sqrt{3}AO = \sqrt{3}$.

Solution 2

Let U be the point in $\mathcal{S}$ vertically above O and let B be the point where AU intersects the circle. (There will be one other point U in $\mathcal{S}$ with UO perpendicular to AC; this point will be vertically below O. By symmetry, the length of UO is the same in either case.)
Join UO and BC.



Since AC is a diameter, then $\angle ABC = 90^\circ$.

Therefore, $\triangle ABC$ is similar to $\triangle AOU$ since each is right-angled and each includes the angle at A.

Thus, $\dfrac{AB}{AC} = \dfrac{AO}{AU}$.

Since U is in $\mathcal{S}$, then $BU = 1$, so $AU = AB + BU = AB + 1$.

Also, $AO = 1$ and $AC = 2$ since the radius of the circle is 1.

Therefore, $\dfrac{AB}{2} = \dfrac{1}{AB+1}$ or $AB^2 + AB - 2 = 0$.

Factoring, we obtain $(AB-1)(AB+2) = 0$. Since $AB > 0$, then $AB = 1$ and so $AU = 2$.

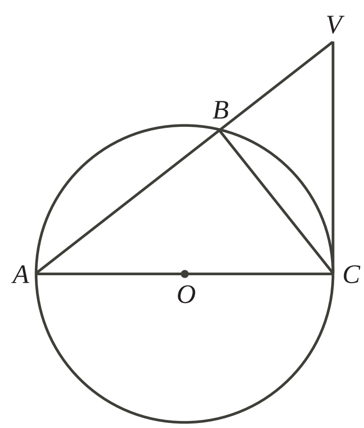
By the Pythagorean Theorem, since $UO > 0$, then $UO = \sqrt{AU^2 - AO^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$.

(b) As in (a), we can choose the point V in $\mathcal{S}$ that is vertically above C.

B is the point where AV intersects the circle. Note that $BV = 1$ by definition.

Join VC and BC.

Since AC is a diameter, then $\angle ABC = 90^\circ$.



Let $VC = x$.

Since $\triangle VCA$ is right-angled at C and $AV > 0$, then by the Pythagorean Theorem, $AV = \sqrt{AC^2 + CV^2} = \sqrt{4 + x^2}$.

Now $\triangle VBC$ is similar to $\triangle VCA$ since both are right-angled and the triangles share a common angle at V.

Since these triangles are similar, then

$$\begin{aligned}
\frac{VB}{VC} &= \frac{VC}{VA} \\
\frac{1}{x} &= \frac{x}{\sqrt{x^2+4}} \\
\sqrt{x^2+4} &= x^2 \\
x^2 + 4 &= x^4 \\
0 &= x^4 - x^2 - 4 \\
0 &= (x^2)^2 - x^2 - 4
\end{aligned}$$

This is a quadratic equation in x^2. By the quadratic formula,

$$x^2 = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

Since x^2 is positive, then $x^2 = \dfrac{1+\sqrt{17}}{2}$.

Since $VC = x$ is positive, then $VC = x = \sqrt{\dfrac{1+\sqrt{17}}{2}}$.

(c) We prove that such a circle does not exist by contradiction.

Suppose that there is a circle Z on which all of the points in $\mathcal{S}$ lie.

We coordinatize the original diagram, putting O at $(0,0)$, C at $(1,0)$, and A at $(-1,0)$.

For every point X in $\mathcal{S}$ that is above AC, there will be a corresponding point Y in $\mathcal{S}$ that is below AC which is the reflection of X in AC.

Therefore, $\mathcal{S}$ is symmetric across the x-axis. Thus, Z is also symmetric across the x-axis and so its centre lies on the x-axis.

Suppose Z has centre $(p,0)$ and radius r.

Then the equation of Z is $(x-p)^2 + y^2 = r^2$.

From (a), the point $(0,\sqrt{3})$ lies on Z. Thus, $p^2 + 3 = r^2$.

Also, the point $W(2,0)$ lies on Z. This point comes from choosing B to coincide with C and extending AB horizontally by 1 unit. Thus, $(2-p)^2 + 0^2 = r^2$ or $p^2 - 4p + 4 = r^2$.

From the equations $p^2 + 3 = r^2$ and $p^2 - 4p + 4 = r^2$, we equate values of r^2 to obtain $p^2 + 3 = p^2 - 4p + 4$ or $4p = 1$ or $p = \frac{1}{4}$.

Thus, $r^2 = p^2 + 3 = \frac{1}{16} + 3 = \frac{49}{16}$ and so $r = \frac{7}{4}$ since $r > 0$.

Therefore, the equation of Z must be $(x - \frac{1}{4})^2 + y^2 = (\frac{7}{4})^2$.

From (b), the point $\left(1, \sqrt{\frac{1+\sqrt{17}}{2}}\right)$ lies on the circle.

Therefore,

$$\begin{aligned}
(1-\tfrac{1}{4})^2 + \left(\sqrt{\tfrac{1+\sqrt{17}}{2}}\right)^2 &= (\tfrac{7}{4})^2 \\
(\tfrac{3}{4})^2 + \tfrac{1+\sqrt{17}}{2} &= \tfrac{49}{16} \\
\tfrac{9}{16} + \tfrac{8}{16} + \tfrac{\sqrt{17}}{2} &= \tfrac{49}{16} \\
\tfrac{\sqrt{17}}{2} &= 2 \\
\sqrt{17} &= 4
\end{aligned}$$

This statement is false, so we have reached a contradiction.

Therefore, our assumption is false and there is no circle on which all of the points in $\mathcal{S}$ lie.

4. (a) First we note that if $x > 0$, then $x + \dfrac{1}{x} > 1$, since if $x \geq 1$, then $x + \dfrac{1}{x} \geq 1 + \dfrac{1}{x} > 1$ and if $0 < x < 1$, then $\dfrac{1}{x} > 1$, so $x + \dfrac{1}{x} > 1$.

We note that $f(x) = x$ is equivalent to

$$\begin{aligned}\left(x+\frac{1}{x}\right)-\left\lfloor x+\frac{1}{x}\right\rfloor &= x\\ \frac{1}{x} &= \left\lfloor x+\frac{1}{x}\right\rfloor\end{aligned}$$

In this last equation, the right side is a positive integer, so the left side is also a positive integer.
Suppose that $\frac{1}{x} = n$ for some positive integer n. Then $x = \frac{1}{n}$.
Therefore, the equation $\frac{1}{x} = \left\lfloor x+\frac{1}{x}\right\rfloor$ is equivalent to the equation $n = \left\lfloor \frac{1}{n}+n\right\rfloor$.
Note that if $n \geq 2$, then $\frac{1}{n} < 1$, so $n < n + \frac{1}{n} < n+1$, which says that $\left\lfloor \frac{1}{n}+n\right\rfloor = n$ so this result is true for all positive integers $n \geq 2$.
Note also that if $n = 1$, then $n + \frac{1}{n} = 2$, so $\left\lfloor \frac{1}{n}+n\right\rfloor \neq n$.
Therefore, if n is a positive integer, then $n = \left\lfloor \frac{1}{n}+n\right\rfloor$ if and only if $n \geq 2$.
Therefore, the solution set of the equation $f(x) = x$ is $x = \frac{1}{n}$, where n is a positive integer with $n \geq 2$.

(b) Suppose that $x = \frac{a}{a+1}$ for some positive integer $a > 1$.

First, we calculate $f(x)$.
Note that

$$x+\frac{1}{x} = \frac{a}{a+1}+\frac{a+1}{a} = \frac{a^2+(a+1)^2}{a(a+1)} = \frac{2a^2+2a+1}{a^2+a} = \frac{2(a^2+a)+1}{a^2+a} = 2+\frac{1}{a^2+a}$$

Since $a > 1$, then $\frac{1}{a^2+a} < \frac{1}{1^2+1} = \frac{1}{2}$ and so $2 < 2 + \frac{1}{a^2+a} < 3$.
Therefore, $\left\lfloor x+\frac{1}{x}\right\rfloor = \left\lfloor 2+\frac{1}{a^2+a}\right\rfloor = 2$.
Thus, if $x = \frac{a}{a+1}$, then $f(x) = \left(x+\frac{1}{x}\right) - \left\lfloor x+\frac{1}{x}\right\rfloor = \left(2+\frac{1}{a^2+a}\right) - 2 = \frac{1}{a^2+a}$.

Second, we show that $x \neq f(x)$.
Note that $x = \frac{a}{a+1} = \frac{a^2}{a(a+1)}$ and $f(x) = \frac{1}{a(a+1)}$, so x and $f(x)$ would be equal if and only if $\frac{a^2}{a(a+1)} = \frac{1}{a(a+1)}$ which is true if and only if $a^2 = 1$.
Since $a > 1$, this is not true, so $x \neq f(x)$.
(Alternatively, we could note that, from (a), $x = f(x)$ if and only if x is of the form $\frac{1}{n}$ for

some positive integer $n > 1$. Here, $x = \dfrac{a}{a+1}$ which is not of this form when $a > 1$, so $x \neq f(x)$.)

Third, we show that $f(x) = f(f(x))$.
We set $y = f(x) = \dfrac{1}{a^2 + a}$.
Since a is a positive integer with $a > 1$, then y is of the form $\dfrac{1}{n}$ for some positive integer n with $n > 2$.
Thus, y is of the form discovered in (a), so $f(y) = y$; in other words, $f(f(x)) = f(x)$.

Therefore, if $x = \dfrac{a}{a+1}$ for some positive integer $a > 1$, then $x \neq f(x)$, but $f(x) = f(f(x))$.

(c) *Solution 1*

We want to find an infinite family of rational numbers u with the properties that

- $0 < u < 1$,
- u, $f(u)$, and $f(f(u))$ are all distinct, and
- $f(f(u)) = f(f(f(u)))$.

We will do this by finding an infinite family of rational numbers u with $0 < u < 1$ with the property that $f(u) = \dfrac{a}{a+1}$ for some positive integer $a > 1$.
In this case, (b) shows that $f(f(u)) = \dfrac{1}{a^2 + a}$ and that $f(f(f(u))) = \dfrac{1}{a^2 + a}$.
Thus, we will have $f(f(u)) = f(f(f(u)))$ and $f(u) \neq f(f(u))$.
As long as we have $u \neq \dfrac{a}{a+1}$ and $u \neq \dfrac{1}{a^2 + a}$, then we will have found a family of rational numbers u with the required properties.
Note that in fact we cannot have $u = \dfrac{1}{a^2 + a}$ because in this case we would have $f(u) = u$ and so we would not have $f(u) = \dfrac{a}{a+1}$.

We now show the existence of an infinite family of rational numbers u with $0 < u < 1$ with $f(u) = \dfrac{a}{a+1}$ for some positive integer $a > 1$.
Let us consider candidate rational numbers $u = \dfrac{b}{b+c}$ with b and c positive integers and $c > 1$.
Since $b + c > b$, then each is a rational number with $0 < u < 1$.
In this case, $u + \dfrac{1}{u} = \dfrac{b}{b+c} + \dfrac{b+c}{b} = \dfrac{b^2 + (b+c)^2}{b(b+c)} = \dfrac{2b^2 + 2bc + c^2}{b^2 + bc} = 2 + \dfrac{c^2}{b^2 + bc}$.
If we suppose further that $c^2 < b^2 + bc$, then $\dfrac{c^2}{b^2 + bc} < 1$ and so $u + \dfrac{1}{u} = 2 + \dfrac{c^2}{b^2 + bc} < 3$, which gives

$$f(u) = \left(u + \frac{1}{u}\right) - \left\lfloor u + \frac{1}{u} \right\rfloor = 2 + \frac{c^2}{b^2 + bc} - 2 = \frac{c^2}{b^2 + bc}$$

We want $f(u)$ to be of the form $\frac{a}{a+1}$. In other words, we want $\frac{c^2}{b^2+bc}$ to be of the form $\frac{a}{a+1}$, which would be true if $b^2+bc-c^2=1$.
Note that if $b^2+bc-c^2=1$, then $c^2=b^2+bc-1<b^2+bc$, so the additional assumption above is included in this equation. Also, if $b^2+bc-c^2=1$, then b and c can have no common divisor larger than 1 so $u=\frac{b}{b+c}$ is irreducible. Combining this with the fact that $c\neq 1$, we see that $\frac{b}{b+c}$ cannot be of the form $\frac{a}{a+1}$.
To summarize so far, if $b^2+bc-c^2=1$ has an infinite family of positive integer solutions (b,c), then the infinite family of rational numbers $u=\frac{b}{b+c}$ has the required properties.

Consider the equation $b^2+bc-c^2=1$.
This is equivalent to the equations $4b^2+4bc-4c^2=4$ and $4b^2+4bc+c^2-5c^2=4$ and $(2b+c)^2-5c^2=4$.
If we let $d=2b+c$, we obtain the equation $d^2-5c^2=4$.
This is a version of Pell's equation. It is known that if such an equation has one positive integer solution, then it has infinitely many positive integer solutions.
Since $d^2-5c^2=4$ has one positive integer solution $(d,c)=(7,3)$, then it has infinitely many positive integer solutions (d,c).
If $d^2=5c^2+4$ and c is odd, then c^2 is odd, so $d^2=5c^2+4$ is odd, which means that d is odd.
If $d^2=5c^2+4$ and c is even, then c^2 is even, so $d^2=5c^2+4$ is even, which means that d is even.
Therefore, if (d,c) satisfies $d^2-5c^2=4$, then d and c have the same parity so $b=\frac{1}{2}(d-c)$ is an integer.
In addition, since $d^2=5c^2+4>c^2$ then $d>c$ which means that $b=\frac{1}{2}(d-c)$ is a positive integer.
Therefore, each positive integer solution (d,c) of the equation $d^2-5c^2=4$ gives a solution (b,c) of the equation $b^2+bc-c^2=1$ which is also a positive integer solution.

Therefore, there exists an infinite family of rational numbers $u=\frac{b}{b+c}$ with the required properties.

Solution 2
As in Solution 1, we want to show the existence of an infinite family of rational numbers u with $0<u<1$ with $f(u)=\frac{a}{a+1}$ for some positive integer $a>1$.
Consider the Fibonacci sequence which has $F_1=1$, $F_2=1$, and $F_n=F_{n-1}+F_{n-2}$ for $n\geq 3$.

Define $u_n = \dfrac{F_{2n-1}}{F_{2n+1}}$ for each positive integer $n \geq 2$.
Note that $0 < F_{2n-1} < F_{2n+1}$ so $0 < u < 1$.
For example, $u_2 = \dfrac{F_3}{F_5} = \dfrac{2}{5}$.
In this case, $f(u_2) = (\frac{2}{5} + \frac{5}{2}) - \lfloor \frac{2}{5} + \frac{5}{2} \rfloor = \frac{29}{10} - \lfloor \frac{29}{10} \rfloor = \frac{9}{10}$, which has the desired properties.
We must show that u_n is not of the form $\dfrac{a}{a+1}$ or of the form $\dfrac{1}{a^2+a}$:

If $\dfrac{F_{2n-1}}{F_{2n+1}} = \dfrac{a}{a+1}$, then $aF_{2n-1} + F_{2n-1} = aF_{2n+1}$ or $F_{2n-1} = a(F_{2n+1} - F_{2n-1})$ or $F_{2n-1} = aF_{2n}$. Since a is a positive integer and $F_{2n} > F_{2n-1}$, this cannot be the case.

If $\dfrac{F_{2n-1}}{F_{2n+1}} = \dfrac{1}{a^2+a}$, then F_{2n+1} is divisible by F_{2n-1}. But F_{j+1} and F_{j-1} never have a common divisor larger than 1, so this cannot be the case. (If F_{j+1} and F_{j-1} have a common divisor larger than 1, then $F_j = F_{j+1} - F_{j-1}$ also has this divisor. We can continue this process using the equation $F_{j-2} = F_j - F_{j-1}$ to show that F_{j-2} also has this divisor, and so on, until we obtain that F_2 and F_1 both have this divisor. Since $F_2 = F_1 = 1$, we have a contradiction.)

In general, note that

$$\begin{aligned} u_n + \frac{1}{u_n} &= \frac{F_{2n-1}}{F_{2n+1}} + \frac{F_{2n+1}}{F_{2n-1}} \\ &= \frac{(F_{2n-1})^2 + (F_{2n+1})^2}{F_{2n-1}F_{2n+1}} \\ &= \frac{(F_{2n-1})^2 + (F_{2n} + F_{2n-1})^2}{F_{2n-1}(F_{2n} + F_{2n-1})} \\ &= \frac{2(F_{2n-1})^2 + 2F_{2n}F_{2n-1} + (F_{2n})^2}{(F_{2n-1})^2 + F_{2n}F_{2n-1}} \\ &= 2 + \frac{(F_{2n})^2}{(F_{2n-1})^2 + F_{2n}F_{2n-1}} \\ &= 2 + \frac{(F_{2n})^2}{F_{2n-1}F_{2n+1}} \end{aligned}$$

It is known that $(F_{2n})^2 - F_{2n-1}F_{2n+1} = -1$ for all positive integers n. (See the end of the solution for a proof of this.)
Set $a_n = (F_{2n})^2$, which is a positive integer.
Then, $u_n + \dfrac{1}{u_n} = 2 + \dfrac{a_n}{a_n+1}$.

Therefore,

$$\begin{aligned}
f(u_n) &= \left(u_n + \frac{1}{u_n}\right) - \left\lfloor u_n + \frac{1}{u_n} \right\rfloor \\
&= \left(2 + \frac{a_n}{a_n+1}\right) - \left\lfloor 2 + \frac{a_n}{a_n+1} \right\rfloor \\
&= \left(2 + \frac{a_n}{a_n+1}\right) - 2 \\
&= \frac{a_n}{a_n+1}
\end{aligned}$$

Therefore, the infinite family of rational numbers u_n has the desired properties.

As a postscript, we prove that $(F_m)^2 - F_{m-1}F_{m+1} = (-1)^{m+1}$ for all positive integers $m \geq 2$.
We prove this result by induction on m.
When $m = 2$, we obtain $(F_2)^2 - F_1F_3 = 1^2 - 1(2) = -1 = (-1)^{2+1}$, as required.
Suppose that the result is true for $m = k$, for some positive integer $k \geq 2$.
That is, suppose that $(F_k)^2 - F_{k-1}F_{k+1} = (-1)^{k+1}$.
Consider $m = k + 1$. Then

$$\begin{aligned}
(F_{k+1})^2 - F_kF_{k+2} &= (F_k + F_{k-1})^2 - F_k(F_k + F_{k+1}) \\
&= (F_k)^2 + 2F_kF_{k-1} + (F_{k-1})^2 - (F_k)^2 - F_kF_{k+1} \\
&= 2F_kF_{k-1} + (F_{k-1})^2 - F_kF_{k+1} \\
&= 2F_kF_{k-1} + (F_{k-1})^2 - F_k(F_k + F_{k-1}) \\
&= 2F_kF_{k-1} + (F_{k-1})^2 - (F_k)^2 - F_kF_{k-1} \\
&= F_kF_{k-1} + (F_{k-1})^2 - (F_k)^2 \\
&= F_{k-1}(F_k + F_{k-1}) - (F_k)^2 \\
&= F_{k-1}F_{k+1} - (F_k)^2 \\
&= (-1)((F_k)^2 - F_{k-1}F_{k+1}) \\
&= (-1)(-1)^{k+1} \qquad \text{(by our inductive assumption)} \\
&= (-1)^{(k+1)+1}
\end{aligned}$$

as required.
Therefore, $(F_m)^2 - F_{m-1}F_{m+1} = (-1)^{m+1}$ for all positive integers $m \geq 2$ by induction, which shows that $(F_{2k})^2 - F_{2k-1}F_{2k+1} = (-1)^{2k+1} = -1$.