The Canadian Mathematical Society La Société mathématique du Canada

The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

Sun Life Financial Canadian Open Mathematics Challenge

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Solutions

Part A

1.
$$\begin{aligned} &-1+2-3+4-5+6-7+8-9+10-11+12-13+14-15+16-17+18 \\ &\quad = (2-1)+(4-3)+(6-5)+(8-7)+(10-9)+(12-11)+ \\ &\qquad (14-13)+(16-15)+(18-17) \\ &\quad = 1+1+1+1+1+1+1+1+1 \\ &\quad = 9 \end{aligned}$$

ANSWER: 9

2. We write 5073 in place value notation as $5 \times 1000 + 7 \times 10 + 3$ or $5 \times 10^3 + 7 \times 10^1 + 3 \times 10^0$.
Thus, if $a = 0$, $b = 3$ and $c = 1$, then the left side $(3 \times 10^a + 5 \times 10^b + 7 \times 10^c)$ equals 5073.
Any other combination of values for a, b and c will not give 5073.
Therefore, $a + b + c = 0 + 3 + 1 = 4$.

(We can show that the only possibility is $a = 0$, $b = 3$ and $c = 1$. We start by noting that the remainder when the right side is divided by 10 is 3, so the remainder when the left side is divided by 10 must also be 3. If $a = 0$ and each of b and c is larger than 0, then the remainder on the left side will be 3. If more than one of a, b and c equals 0, then we can see by trying the possibilities that the remainder on the left side cannot be 3.
Therefore, $a = 0$ and b and c are positive.
We can then subtract 3 from both sides and divide by 10 to obtain the new equation

$$5 \times 10^{b-1} + 7 \times 10^{c-1} = 507$$

and repeat the argument to show that $c = 1$ and then $b = 3$.)

ANSWER: 4

3. *Solution 1*
Suppose that Soroosh has d dimes.
Since he has 10 coins, then he has $10 - d$ quarters.
The value of the dimes is $10d$ cents and the value of the quarters is $25(10 - d)$ cents.
Since we want the value of the dimes to be larger than the value of the quarters, then

$$\begin{aligned} 10d &> 25(10-d) \\ 10d &> 250 - 25d \\ 35d &> 250 \\ 7d &> 50 \\ d &> \tfrac{50}{7} = 7\tfrac{1}{7} \end{aligned}$$

Since d is an integer, then $d \geq 8$, so the smallest possible number of dimes is 8.

Solution 2

We proceed by systematic trial and error.

If Soroosh has 4 quarters and 6 dimes, then the quarters are worth $4 \times 25 = 100$ cents and the dimes are worth $6 \times 10 = 60$ cents. (Any smaller number of dimes than 6 makes the value of the dimes smaller and the value of quarters larger, so the number of dimes must be greater than 6.)

If Soroosh has 3 quarters and 7 dimes, then the quarters are worth $3 \times 25 = 75$ cents and the dimes are worth $7 \times 10 = 70$ cents.

If Soroosh has 2 quarters and 8 dimes, then the quarters are worth $2 \times 25 = 50$ cents and the dimes are worth $8 \times 10 = 80$ cents.

Therefore, the smallest number of dimes for which the value of the dimes is greater than the value of the quarters is 8.

Answer: 8

4. *Solution 1*

From the given conditions, we want $15(12) = 180$ to be divisible by n, and $15n$ to be divisible by 12, and $12n$ to be divisible by 15.

For $15n$ to be divisible by 12, then $15n$ is a multiple of 12, or $15n = 12m$ for some positive integer m. Simplifying, we see that $5n = 4m$.

Since the right side is divisible by 4, then the left side must be divisible by 4, so n must be divisible by 4.

For $12n$ to be divisible by 15, we must have $12n = 15k$ for some positive integer k. Simplifying, we see that $4n = 5k$.

Since the right side is a multiple of 5, then the left side must be a multiple of 5, so n must be a multiple of 5.

Therefore, n must be a multiple of 4 and a multiple of 5.

This tells us that n must be a multiple of 20.

Since we want n to be as small as possible, then we try $n = 20$, since this is the smallest positive multiple of 20.

If $n = 20$, then it is true that $15(12) = 180$ is divisible by 20, and $15(20) = 300$ is divisible by 12, and $12(20) = 240$ is divisible by 15.

Thus, the smallest possible value of n is 20.

(We could instead have started with the condition 180 is divisible by n, listed the positive divisors of 180, and then tried these divisors starting from the smallest until we found a divisor that satisfied the other two conditions.)

Solution 2

First, we note that the prime factorizations of 12 and 15 are $12 = 2^2 \cdot 3$ and $15 = 3 \cdot 5$.

Since $12 \mid 15n$, then n must contain a factor of 2^2 since 12 does and 15 is not divisible by 2.

Since $15 \mid 12n$, then n must contain a factor of 5 since 15 does and 12 does not.

Since $n \mid 12(15) = 2^2 \cdot 3^2 \cdot 5$, then n cannot contain more than 2 factors of 2 and 1 factor of 5, since 12(15) contains only 2 factors of 2 and 1 of 5.

Therefore, to make n as small as possible, n must be exactly $2^2 \cdot 5 = 20$.

(Can you find the other values of n that work?)

Answer: 20

5. *Solution 1*

We represent the sequence of islands that Maya visits as a sequence of letters starting with A. Since she makes 20 bridge crossings, then she visits 21 islands in total, so the sequence contains 21 letters.

If Maya is on island A or on island C, then the next island that she visits must be island B, since it is the only island connected to A and the only island connected to C.

If Maya is on island B, then Maya has two choices: cross to island A or cross to island C.

Since she starts at island A, then the second letter in the sequence must be B, since she must cross to island B.

The third letter can be either A or C, as she has a choice from island B.

Once on island A or C, she must cross back to island B, so the fourth letter is B.

She is thus in the same situation as she was after her first crossing, and so the pattern continues. In other words, the letters in odd positions in the sequence, starting at the third, can be either A or C, and the letters in the even positions must be B.

We can represent the sequence then as follows:

$$A\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}\ B\ {A \atop C}$$

Thus, there are 10 positions in the sequence where Maya has 2 choices and the rest of the positions are fixed.

Thus, there are $2^{10} = 1024$ possible sequences.

Solution 2

Define S_n to be the number of sequences starting at island A with n crossings. We want to determine S_{20}.

Note that $S_2 = 2$ (A to B to A, and A to B to C are the possible routes).

First, we note that islands A and C are interchangeable, since we could switch the labels and there would be no structural difference to the diagram.

Thus, the number of sequences of a given length starting at A is the same as the number of sequences of the same length starting at C.

Suppose that Maya is going to make a journey with t crossings, where t is an even integer with $t \geq 4$. There are S_t such sequences.

After two crossings, Maya would be at either island A or island C and would have $t-2$ crossings remaining.

But starting at either island A or C, there are S_{t-2} sequences that she could follow.

Therefore, $S_t = S_{t-2} + S_{t-2} = 2S_{t-2}$.

Now,

$$S_{20} = 2S_{18} = 2(2S_{16}) = 2^2 S_{16} = 2^3 S_{14} = \cdots = 2^9 S_2 = 2^9(2) = 2^{10} = 1024$$

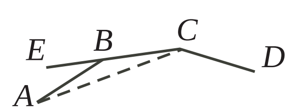
Therefore, there are 1024 possible sequences.

Answer: $2^{10} = 1024$

6. *Solution 1*

Suppose that the polygon has n sides.

Extend CB outside of the polygon. Since the sum of the exterior angles in a polygon is always 360°, then $\angle ABE = \left(\frac{360}{n}\right)^\circ$, since there will be n equal exterior angles.



Thus, $\angle ABC = 180^\circ - \left(\frac{360}{n}\right)^\circ$ and this will also be the measure of $\angle BCD$, since the polygon is regular.

Since the polygon is regular, then $AB = BC$, so $\triangle ABC$ is isosceles, which means that we have $\angle BAC = \angle BCA$.

Therefore,

$$\angle BCA = \frac{1}{2}(180^\circ - \angle ABC) = \frac{1}{2}\left(180^\circ - \left(180^\circ - \left(\frac{360}{n}\right)^\circ\right)\right) = \left(\frac{180}{n}\right)^\circ$$

But $\angle BCD = \angle BCA + \angle ACD$, so

$$\begin{aligned} 180^\circ - \left(\frac{360}{n}\right)^\circ &= \left(\frac{180}{n}\right)^\circ + 120^\circ \\ 60 &= \frac{540}{n} \\ n &= 9 \end{aligned}$$

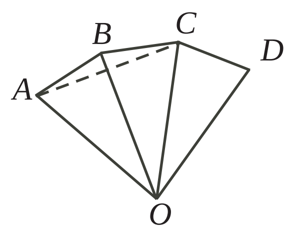
Therefore, the polygon has 9 sides.

Solution 2

Suppose that the polygon has n sides.

Let O be the centre of the polygon. Join O to each of A, B, C, and D.

Since the polygon is regular, then the angle subtended at O by each of the n sides will be equal, and these angles all add to 360°.



Since there are n equal central angles, then $\angle AOB = \angle BOC = \angle COD = \left(\frac{360}{n}\right)^\circ$.

This also tells us that $\angle AOC = \angle AOB + \angle BOC = 2\left(\frac{360}{n}\right)^\circ = \left(\frac{720}{n}\right)^\circ$.

Since the polygon is regular, then $OA = OC = OD$, which tells us that $\triangle AOC$ and $\triangle COD$ are both isosceles.

Thus,

$$\angle ACO = \frac{1}{2}(180^\circ - \angle AOC) = \frac{1}{2}\left(180^\circ - \left(\frac{720}{n}\right)^\circ\right) = 90^\circ - \left(\frac{360}{n}\right)^\circ$$

and

$$\angle DCO = \frac{1}{2}(180^\circ - \angle COD) = \frac{1}{2}\left(180^\circ - \left(\frac{360}{n}\right)^\circ\right) = 90^\circ - \left(\frac{180}{n}\right)^\circ$$

Now, $\angle ACD = \angle ACO + \angle DCO$, so

$$\begin{aligned} 120^\circ &= 90^\circ - \left(\frac{360}{n}\right)^\circ + 90^\circ - \left(\frac{180}{n}\right)^\circ \\ \frac{540}{n} &= 60 \\ n &= 9 \end{aligned}$$

Therefore, the polygon has 9 sides.

Answer: 9

7. Using the rules for manipulating logarithms and trigonometric functions,

$$\begin{aligned}
\log_2(-3\sin\theta) &= 2\log_2(\cos\theta)+1\\
\log_2(-3\sin\theta) &= \log_2(\cos^2\theta)+\log_2 2\\
\log_2(-3\sin\theta) &= \log_2(2\cos^2\theta)\\
2^{\log_2(-3\sin\theta)} &= 2^{\log_2(2\cos^2\theta)}\\
-3\sin\theta &= 2\cos^2\theta\\
-3\sin\theta &= 2(1-\sin^2\theta) \qquad (\text{since } \cos^2\theta+\sin^2\theta=1)\\
2\sin^2\theta-3\sin\theta-2 &= 0\\
(2\sin\theta+1)(\sin\theta-2) &= 0
\end{aligned}$$

Therefore, $\sin\theta = -\frac{1}{2}$ or $\sin\theta = 2$.
The second possibility is inadmissible, so $\sin\theta = -\frac{1}{2}$.
Since $0^\circ \le \theta \le 360^\circ$ and $\sin\theta = -\frac{1}{2}$, then $\theta = 210^\circ$ or $\theta = 330^\circ$.
But, we also need $\cos\theta > 0$ to satisfy the domains of the logarithms in the original equation.
Therefore, $\theta = 210^\circ$ is inadmissible (since it is in the third quadrant and $\cos\theta < 0$), but $\theta = 330^\circ$ is the admissible (since it is in the fourth quadrant and $\cos\theta > 0$).
Checking, $\sin(330^\circ) = -\frac{1}{2}$ and $\cos(330^\circ) = \frac{\sqrt{3}}{2}$, so the left side of the original equation equals $\log_2\left(\frac{3}{2}\right)$ and the right side equals $2\log_2\left(\frac{\sqrt{3}}{2}\right)+1 = \log_2\left(\frac{3}{4}\right)+\log_2(2) = \log_2\left(\frac{3}{2}\right)$, as required.
Therefore, $\theta = 330^\circ$.

Answer: 330°

8. We examine three cases: $b = c$, $b > c$ and $b < c$.
Note that, in any of these cases, we have $a! > 4(b!)$ and $a! > 10(c!)$ so $a > b$ and $a > c$.

Case 1: $b = c$
Here, the equation becomes $a! = 14(b!)$ or $\dfrac{a!}{b!} = 14$ or $a(a-1)\cdots(b+2)(b+1) = 14$.
The expression on the left side is a single integer (if $a = b+1$) or the product of 2 or more consecutive integers.
Since $14 = 2(7)$, then 14 cannot be written as the product of two or more consecutive integers.
Therefore, the expression on the left must be a single integer.
Therefore, $a = b+1 = 14$, so $b = c = 13$.
Thus, the only solution in this case is $(a,b,c) = (14,13,13)$.

Case 2: $b > c$
Dividing both sides by $b!$, the equation becomes $\dfrac{a!}{b!} = 4 + \dfrac{10(c!)}{b!}$ or

$$a(a-1)\cdots(b+2)(b+1) = 4 + \frac{10}{b(b-1)\cdots(c+2)(c+1)}$$

Since the left side is an integer, then the right side must be an integer. Thus,

$$\frac{10}{b(b-1)\cdots(c+2)(c+1)}$$

is an integer, which means that 10 is divisible by $b(b-1)\cdots(c+2)(c+1)$, which is again a single integer (if $b = c+1$) or the product of 2 or more consecutive integers, each of which is at least 2 (since $c \geq 1$).

As in Case 1, the only possibility is that the denominator is one of the single integers 10, 5 and 2.

The possibilities are thus $b = c+1 = 10$ (whence $b = 10$ and $c = 9$), $b = c+1 = 5$ (whence $b = 5$ and $c = 4$), or $b = c+1 = 2$ (whence $b = 2$ and $c = 1$).

If $b = 10$ and $c = 9$, the right side of the initial equation becomes $4(10!)+10(9!)$ or $4(10!)+(10!)$, which equals $5(10!)$. This number is not a factorial because it is bigger than 10! and less than 11!. There is thus no possible value for a.

If $b = 5$ and $c = 4$, the right side of the initial equation becomes $4(5!) + 10(4!)$, which equals $480 + 240 = 720 = 6!$, and so $a = 6$.

If $b = 2$ and $c = 1$, the right side of the initial equation becomes $4(2!) + 10(1!) = 8 + 10 = 18$. This number is not a factorial because it is bigger than 3! and less than 4!. There is thus no possible value for a.

Therefore, the only solution in this case is $(a, b, c) = (6, 5, 4)$.

Case 3: $b < c$

Dividing both sides by $c!$, the equation becomes $\frac{a!}{c!} = \frac{4(b!)}{c!} + 10$ or

$$a(a-1)\cdots(c+2)(c+1) = \frac{4}{c(c-1)\cdots(b+2)(b+1)} + 10$$

Since the left side is an integer, then the right side must be an integer. Thus,

$$\frac{4}{c(c-1)\cdots(b+2)(b+1)}$$

is an integer, which means that 4 is divisible by $c(c-1)\cdots(b+2)(b+1)$, which is again a single integer (if $c = b+1$) or the product of 2 or more consecutive integers, each of which is at least 2 (since $b \geq 1$).

As in Case 2, the only possibility is that the denominator is one of the single integers 4 and 2.

The possibilities are thus $c = b+1 = 4$ (whence $c = 4$ and $b = 3$) or $c = b+1 = 2$ (whence $c = 2$ and $b = 1$).

If $c = 4$ and $b = 3$, the right side of the initial equation becomes $4(3!) + 10(4!)$ which equals $24 + 240 = 264$. This number is not a factorial because it is bigger than 5! and less than 6!. There is thus no possible value for a.

If $c = 2$ and $b = 1$, the right side of the initial equation becomes $4(1!) + 10(2!)$ which equals

$4 + 20 = 24 = 4!$, and so $a = 4$.
Therefore, the only solution in this case is $(a, b, c) = (4, 1, 2)$.

Therefore, the three solutions are $(a, b, c) = (14, 13, 13)$, $(6, 5, 4)$, and $(4, 1, 2)$.

Answer: $(a, b, c) = (14, 13, 13)$, $(6, 5, 4)$, and $(4, 1, 2)$

Part B

1. (a) *Solution 1*

By the Pythagorean Theorem, since $AC > 0$,

$$AC = \sqrt{CB^2 - AB^2} = \sqrt{15^2 - 9^2} = \sqrt{225 - 81} = \sqrt{144} = 12$$

Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AB)(AC) = \frac{1}{2}(9)(12) = 54$.

Solution 2

Since $\triangle ABC$ is right-angled at A and $AB : CB = 9 : 15 = 3 : 5$, then $\triangle ABC$ is similar to a $3 : 4 : 5$ triangle.
Therefore, $AC = \frac{4}{3}AB = \frac{4}{3}(9) = 12$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AB)(AC) = \frac{1}{2}(9)(12) = 54$.

(b) From (a), $AC = 12$.
Since the area of $\triangle CDB$ is 84, then $\frac{1}{2}(DB)(AC) = 84$ or $\frac{1}{2}(DB)(12) = 84$ or $6(DB) = 84$.
Therefore, $DB = 14$ and so $DA = DB - AB = 14 - 9 = 5$.
Lastly, by the Pythagorean Theorem, since $CD > 0$, we have

$$CD = \sqrt{DA^2 + AC^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

(c) Since the area of $\triangle PQR$ is 300, then $\frac{1}{2}(QR)(PT) = 300$ or $\frac{1}{2}(25)(PT) = 300$ or $25(PT) = 600$ or $PT = 24$.
By the Pythagorean Theorem, since $QT > 0$,

$$QT = \sqrt{PQ^2 - PT^2} = \sqrt{25^2 - 24^2} = \sqrt{625 - 576} = \sqrt{49} = 7$$

Thus, $TR = QR - QT = 25 - 7 = 18$.
In $\triangle PTR$, we now have $PT = 24$, $\angle PTR = 90^\circ$, and $TR = 18$.
Lastly, by the Pythagorean Theorem, since $PR > 0$, we have

$$PR = \sqrt{PT^2 + TR^2} = \sqrt{24^2 + 18^2} = \sqrt{576 + 324} = \sqrt{900} = 30$$

(The given diagram implies that T, the foot of the altitude from P to QR, lies between Q and R, although the problem does not explicitly state this. If this implied restriction is removed, there is a second case with $PR = 40$, $\angle PQR$ obtuse, and T to the left of Q.)

2. (a) The line through points Q and M has slope $\dfrac{7-1}{4-19} = \dfrac{6}{-15} = -\dfrac{2}{5}$ and so has equation $y - 7 = -\frac{2}{5}(x - 4)$ or $y = -\frac{2}{5}x + \frac{43}{5}$.

(b) *Solution 1*

The midpoint, N, of PQ has coordinates $\left(\frac{1}{2}(7 + 19), \frac{1}{2}(13 + 1)\right) = (13, 7)$.

The line through points R and N has slope $\dfrac{7-1}{13-1} = \dfrac{6}{12} = \dfrac{1}{2}$ and so has equation $y - 1 = \frac{1}{2}(x-1)$ or $y = \frac{1}{2}x + \frac{1}{2}$.
At the point of intersection of $y = -\frac{2}{5}x + \frac{43}{5}$ and $y = \frac{1}{2}x + \frac{1}{2}$, the values of y are equal so:

$$\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{2} &= -\tfrac{2}{5}x + \tfrac{43}{5} \\ \tfrac{5}{10}x + \tfrac{4}{10}x &= \tfrac{86}{10} - \tfrac{5}{10} \\ \tfrac{9}{10}x &= \tfrac{81}{10} \\ x &= 9 \end{aligned}$$

Since the x-coordinate of G is 9, then the y coordinate is $\frac{1}{2}(9) + \frac{1}{2} = \frac{10}{2} = 5$, so the coordinates of G are $(9,5)$.

Solution 2
Point G is the intersection of two of the medians of $\triangle PQR$, and so is the centroid of $\triangle PQR$. (In fact, all three medians will pass through G.)
The coordinates of the centroid are the averages of the coordinates of the three vertices.
Thus, the coordinates of G are $\left(\frac{1}{3}(7+1+19), \frac{1}{3}(13+1+1)\right) = (9,5)$.

(c) *Solution 1*
The slope of PR is $\dfrac{13-1}{7-1} = \dfrac{12}{6} = 2$.
Since QF is perpendicular to PR, then its slope is the negative reciprocal of 2, or $-\frac{1}{2}$.
Thus, the line through Q and F has equation $y - 1 = -\frac{1}{2}(x - 19)$ or $y = -\frac{1}{2}x + \frac{21}{2}$.
The slope of PQ is $\dfrac{13-1}{7-19} = \dfrac{12}{-12} = -1$.
Since RT is perpendicular to PQ, then its slope is the negative reciprocal of -1, or 1.
Thus, the line through R and T has equation $y - 1 = 1(x-1)$ or $y = x$.
At the point of intersection of these lines, the values of y are equal so:

$$\begin{aligned} x &= -\tfrac{1}{2}x + \tfrac{21}{2} \\ \tfrac{3}{2}x &= \tfrac{21}{2} \\ x &= 7 \end{aligned}$$

Since the x-coordinate of H is 7, then the y coordinate is also 7, since H lies on the line $y = x$.
Thus, the coordinates of H are $(7,7)$.

Solution 2
The three altitudes of $\triangle PQR$ all pass through H.
Since side QR of $\triangle PQR$ is horizontal, then the altitude from P must be vertical.
Since the x-coordinate of Q is 7, then the equation of the altitude through P is $x = 7$.
We can then determine the equation of a second altitude, say the altitude through R and

T as in Solution 1, to be $y = x$.
Therefore, point H lies at the intersection of $y = x$ and $x = 7$, which is the point $(7, 7)$.

(d) The distance between $O(0, 0)$ and G is $\sqrt{(9-0)^2 + (5-0)^2} = \sqrt{81+25} = \sqrt{106}$.
The distance between O and H is $\sqrt{(7-0)^2 + (7-0)^2} = \sqrt{49+49} = \sqrt{98}$.
Since $\sqrt{98} < \sqrt{106}$, then H is closer to the origin than G.

3. (a) To find all real fixed points, we need to solve the equation $f(c) = c$.
Since $f(x) = x^2 - 2$, we solve $c^2 - 2 = c$ or $c^2 - c - 2 = 0$.
Thus, $(c-2)(c+1) = 0$, so the real fixed points are $c = 2$ and $c = -1$.

(b) *Solution 1*

Suppose that $g(x) = ax^3 + bx^2 + dx + e$ for some real coefficients a, b, d, e with $a \neq 0$ (since $g(x)$ is cubic). Suppose also that f and g commute (that is, $f(g(x)) = g(f(x))$ for all real numbers x). Now,

$$\begin{aligned} f(g(x)) &= f(ax^3 + bx^2 + dx + e) \\ &= (ax^3 + bx^2 + dx + e)^2 - 2 \\ &= a^2x^6 + b^2x^4 + d^2x^2 + e^2 + 2abx^5 + 2adx^4 + 2aex^3 + 2bdx^3 + 2bex^2 + 2dex - 2 \\ &= a^2x^6 + 2abx^5 + (b^2 + 2ad)x^4 + (2ae + 2bd)x^3 + (d^2 + 2be)x^2 + 2dex + (e^2 - 2) \end{aligned}$$

and

$$\begin{aligned} g(f(x)) &= g(x^2 - 2) \\ &= a(x^2-2)^3 + b(x^2-2)^2 + d(x^2-2) + e \\ &= a(x^6 - 6x^4 + 12x^2 - 8) + b(x^4 - 4x^2 + 4) + d(x^2 - 2) + e \\ &= ax^6 + (-6a + b)x^4 + (12a - 4b + d)x^2 + (-8a + 4b - 2d + e) \end{aligned}$$

Since $f(g(x)) = g(f(x))$ for all real numbers x, then the coefficients on the left side must equal the coefficients on the right side.
Therefore,

$$\begin{aligned} a^2 &= a & (1) \\ 2ab &= 0 & (2) \\ b^2 + 2ad &= -6a + b & (3) \\ 2ae + 2bd &= 0 & (4) \\ d^2 + 2be &= 12a - 4b + d & (5) \\ 2de &= 0 & (6) \\ e^2 - 2 &= -8a + 4b - 2d + e & (7) \end{aligned}$$

From (1), $a^2 - a = 0$ or $a(a-1) = 0$ and so $a = 1$ or $a = 0$. Since $a \neq 0$, then $a = 1$.
Substituting $a = 1$ into (2), we obtain $2b = 0$ or $b = 0$.
Substituting $a = 1$ and $b = 0$ into (3), we obtain $0 + 2(1)d = -6(1) + 0$ or $2d = -6$, so $d = -3$.
Substituting $d = 3$ into (6), we obtain $-6e = 0$ so $e = 0$.
We can check that $a = 1$, $b = 0$, $d = -3$, and $e = 0$ satisfy equations (4), (5) and (7).
Therefore, $g(x) = 1x^3 + 0x^2 + (-3)x + 0 = x^3 - 3x$ is the only cubic polynomial that commutes with $f(x)$.
(We can check by expanding that $(x^3 - 3x)^2 - 2 = (x^2 - 2)^3 - 3(x^2 - 2)$.)

Solution 2
Suppose that $g(x) = ax^3 + bx^2 + dx + e$ for some real coefficients a, b, d, e with $a \neq 0$ (since $g(x)$ is cubic). Suppose also that f and g commute (that is, $f(g(x)) = g(f(x))$ for all real numbers x). Now,

$$\begin{aligned} f(g(x)) &= f(ax^3 + bx^2 + dx + e) \\ &= (ax^3 + bx^2 + dx + e)^2 - 2 \end{aligned}$$

and

$$\begin{aligned} g(f(x)) &= g(x^2 - 2) \\ &= a(x^2 - 2)^3 + b(x^2 - 2)^2 + d(x^2 - 2) + e \end{aligned}$$

Since $f(g(x)) = g(f(x))$ for all real numbers x, then the coefficients on the left side must equal the coefficients on the right side when expanded.
On the left side, the only term involving x^6 will come from squaring the term ax^3, so the coefficient of x^6 is a^2.
On the right side, the only term involving x^6 comes from $a(x^2 - 2)^3$; since the coefficient of x^6 in $(x^2 - 2)^3$ is 1, then the coefficient of x^6 on the right side is a.
Therefore, $a^2 = a$ or $a^2 - a = 0$ or $a(a-1) = 0$ and so $a = 1$ or $a = 0$. Since $a \neq 0$, then $a = 1$.
When the expansion on the right side is done, there will be only even powers of x.
Thus, the left side cannot contain any odd powers of x.
When the left side is expanded, we will obtain a term $2abx^5$. Thus, $2ab = 0$. Since $a = 1$, then $b = 0$.
Therefore, we have

$$(x^3 + dx + e)^2 - 2 = (x^2 - 2)^3 + d(x^2 - 2) + e$$

On the left side, the only way to obtain an x^4 term is by multiplying x^3 and dx, so the x^4 term on the left side is $2dx^4$. On the right side, the only x^4 term is from the expansion of

$(x^2-2)^3$ and so is $3(-2)(x^2)^2 = -6x^4$.

Comparing coefficients, $2d = -6$ or $d = -3$.

Since the right side contains no term involving x^1, then the coefficient of x on the left side is 0.

When $(x^3 - 3x + e)^2 - 2$ is expanded, the term involving x^1 will be $2(-3x)e = -6ex$ and so $-6e = 0$ or $e = 0$.

Therefore, $g(x) = 1x^3 + 0x^2 + (-3)x + 0 = x^3 - 3x$ is the only cubic polynomial that commutes with f.

(We can check by expanding that $(x^3 - 3x)^2 - 2 = (x^2 - 2)^3 - 3(x^2 - 2)$.)

(c) We prove the desired result by contradiction.

Suppose that $q(x)$ has a real fixed point c; that is, suppose that $q(c) = c$.

Since p and q commute, then $p(q(x)) = q(p(x))$ for all real numbers x.

In particular, $p(q(c)) = q(p(c))$.

Since $q(c) = c$, this equation becomes $p(c) = q(p(c))$.

Since the equation that we are given is true for all real numbers x, then it is true for $x = c$, so

$$\begin{aligned} 2\left[q(p(c))\right]^4 + 2 &= [p(c)]^4 + [p(c)]^3 \\ 2\left[p(c)\right]^4 + 2 &= [p(c)]^4 + [p(c)]^3 \qquad \text{(from above)} \\ [p(c)]^4 - [p(c)]^3 &= -2 \end{aligned}$$

Define $u = p(c)$. Note that u is a real number since c is a real number and $p(x)$ has real coefficients.

To arrive at our contradiction, we show that there are no real numbers u for which $u^4 - u^3 = -2$.

We do this by looking at three cases: $u \geq 1$, $u \leq 0$, and $0 < u < 1$.

If $u \geq 1$, then $u^4 = u(u^3) \geq 1(u^3) = u^3$ so $u^4 - u^3 \geq 0$, which means that $u^4 - u^3 \neq -2$.

If $u \leq 0$, then $u^3 \leq 0$, which means that $u^4 - u^3 \geq u^4 \geq 0$, so $u^4 - u^3 \neq -2$.

If $0 < u < 1$, then $u^4 > 0$ and $u^3 < 1$, and so $-u^3 > -1$. Thus, $u^4 - u^3 > 0 + (-1) = -1$, so $u^4 - u^3 \neq -2$.

In all cases, $u^4 - u^3 \neq -2$.

This is a contradiction, since we have determined that $[p(c)]^4 - [p(c)]^3 = -2$.

Therefore, our original assumption must be incorrect, so $q(x)$ cannot have a real fixed point.

4. (a) We want to find all positive integers a for which the smallest positive s with the property that a divides into $1 + 2 + 3 + \cdots + s$ is $s = 8$.

Note that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$. (We use the term *triangular number* to

mean a positive integer of the form $1+2+3+\cdots+s$.) The previous triangular numbers are $1,3,6,10,15,21,28$. Therefore, we want to find all positive integers a that are divisors of 36 but not of any of $1,3,6,10,15,21,28$.
The divisors of 36 are $1,2,3,4,6,9,12,18,36$.
The divisors $1,2,3,4,6$ each divide into at least one of 6 and 28, so $f(1)\neq 8$, $f(2)\neq 8$, $f(3)\neq 8$, $f(4)\neq 8$, and $f(6)\neq 8$.
The divisors $9,12,18,36$ do not divide into any triangular number smaller than 36.
Therefore, the complete solution to $f(a)=8$ is $a=9,12,18,36$.

(b) For m a positive integer, we define $T(m)=1+2+\cdots+m=\frac{1}{2}m(m+1)$. ($T(m)$ is the mth triangular number.)
First, we show that if $T(z)$ is a multiple of w, then $f(w)\leq z$:

> We know that $f(w)$ is the smallest integer m for which $1+2+3+\cdots+m=T(m)$ is a multiple of w.
> Suppose that $T(z)$ is a multiple of w.
> If z is the smallest positive integer with this property, then $f(w)=z$; otherwise, z is not the smallest such positive integer, so $f(w)<z$.
> In either case, $f(w)\leq z$.

Next, we show that if y is an odd positive integer with $y>1$, then $f(y)\leq y-1$:

> Suppose that $y=2Y+1$ for some positive integer Y.
> Then $T(y-1)=\frac{1}{2}(y-1)y=\frac{1}{2}(2Y)(2Y+1)=Y(2Y+1)=Yy$.
> Thus, $T(y-1)$ is a multiple of y, and so by the first fact above, $f(y)\leq y-1$.

Next, we show that $f(2^a)=2^{a+1}-1$ for every positive integer a:

> Suppose that $f(2^a)=m$.
> Then $\frac{1}{2}m(m+1)$ is a multiple of 2^a, so $\frac{1}{2}m(m+1)=q2^a$ for some positive integer q or $2^{a+1}q=m(m+1)$.
> Since one of m and $m+1$ is even and the other is odd, then the even one of these must contain at least $a+1$ factors of 2 and so must be at least 2^{a+1}.
> The smallest m for which this is possible is $m=2^{a+1}-1$ which makes $m+1=2^{a+1}$.
> This tells us that $f(2^a)\geq 2^{a+1}-1$.
> But $T(2^{a+1}-1)=\frac{1}{2}(2^{a+1}-1)2^{a+1}=2^a(2^{a+1}-1)$, which is divisible by 2^a, so $f(2^a)\leq 2^{a+1}-1$.
> Therefore, $f(2^a)=2^{a+1}-1$.

We can now look at $f(b+1)-f(b)$ when $b=2^a-1$ for some positive integer a.
Note that b is odd.
In this case,

$$f(b+1)-f(b)=f(2^a)-f(2^a-1)=2^{a+1}-1-f(2^a-1)\geq 2^{a+1}-1-(2^a-2)=2^a+1$$

If $a \geq 11$, then $2^a + 1 \geq 2049 > 2009$.

Therefore, if $b = 2^a - 1$ and a is a positive integer with $a \geq 11$, then $f(b+1) - f(b) > 2009$, so there are infinitely many odd positive integers b for which $f(b+1) - f(b) > 2009$.

(c) From (a), we know that $f(c) = f(c+3)$ has a solution, namely $c = 9$, since $f(9) = 8$ and $f(12) = 8$.

We show that $k = 3$ is the smallest possible value of k by showing that the equations

$$f(c) = f(c+1) \qquad \text{and} \qquad f(c) = f(c+2)$$

are not satisfied by any odd positive integer c.

If a and b are positive integers, we use the notation "$a \mid b$" to mean that b is divisible by a (in other words, b is a multiple of a or equivalently a divides b).

Case 1: $f(c) = f(c+1)$

Suppose that $f(c) = f(c+1) = m$ for some odd positive integer c.

Then $c \mid T(m)$ and $T(m) = \frac{1}{2}m(m+1)$; say, $\frac{1}{2}m(m+1) = qc$ for some positive integer q.

Since c is odd, then $m \leq c - 1$ by (b), so $qc = \frac{1}{2}m(m+1) \leq \frac{1}{2}(c-1)(c)$ which tells us that $q \leq \frac{1}{2}(c-1)$.

But if $f(c+1) = m$ as well, then $c+1 \mid T(m)$ and $T(m) = qc$, so $c+1 \mid qc$.

Since c and $c+1$ are consecutive integers, then $\gcd(c, c+1) = 1$.

(This is true since if d is a positive common divisor of c and $c+1$, then d divides into their difference (which equals 1), so d itself must equal 1.)

Since $c+1 \mid qc$, and q and c are positive integers, and $\gcd(c, c+1) = 1$, then $c+1 \mid q$, so $q \geq c+1$.

But $q \leq \frac{1}{2}(c-1)$, which is a contradiction, since we cannot have $q \geq c+1 > \frac{1}{2}(c-1) \geq q$.

Thus, $f(c) = f(c+1)$ has no odd solutions.

Case 2: $f(c) = f(c+2)$

Suppose that $f(c) = f(c+2) = m$ for some odd positive integer c.

Then $c \mid T(m)$; say, $\frac{1}{2}m(m+1) = qc$ for some positive integer q.

Since $m \leq c - 1$ by (c), then $qc \leq \frac{1}{2}(c-1)(c)$ which tells us that $q \leq \frac{1}{2}(c-1)$.

But if $f(c+2) = m$ as well, then $c+2 \mid T(m)$ and $T(m) = qc$, so $c+2 \mid qc$.

Since c and $c+2$ are odd integers, then $\gcd(c, c+2) = 1$.

(This is true since if d is a positive common divisor of c and $c+2$, then c divides into their difference (which equals 2), so d itself must equal 1 or 2. But both c and $c+2$ are odd, so d is odd, so $d = 1$.)

Since $c+2 \mid qc$, and q and c are positive integers, and $\gcd(c, c+2) = 1$, then $c+2 \mid q$, so $q \geq c+2$.

But $q \leq \frac{1}{2}(c-1)$, so we have a contradiction as above.

Thus, $f(c) = f(c+2)$ has no odd solutions.

Therefore, $k = 3$ is the smallest positive integer k for which $f(c) = f(c+k)$ has solutions with c odd, since if $k = 1$ or $k = 2$, there are no solutions, and there is at least 1 solution for $k = 3$.