The Canadian Mathematical Society — La Société mathématique du Canada

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in collaboration with

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

presents the

Sun Life Financial Canadian Open Mathematics Challenge

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Sun Life Financial

Solutions

Part A

1. *Solution 1*

   Adding the two equations gives $(2x + y) + (x + 2y) = 13 + 11$ or $3x + 3y = 24$.
   Thus, $x + y = \frac{1}{3}(24) = 8$.

   Solution 2

   We solve for x first, then y.
   Since $2x + y = 13$, then $4x + 2y = 26$.
   Subtracting the second equation from this, we obtain $(4x+2y)-(x+2y) = 26-11$ or $3x = 15$ or $x = 5$.
   Substituting into the original first equation, we obtain $2(5) + y = 13$ or $y = 13 - 10 = 3$.
   Thus, $x + y = 5 + 3 = 8$.

   Solution 3

   We solve for y first, then x.
   Since $x + 2y = 11$, then $2x + 4y = 22$.
   Subtracting the first equation from this, we obtain $(2x + 4y) - (2x + y) = 22 - 13$ or $3y = 9$ or $y = 3$.
   Substituting into the original second equation, we obtain $x + 2(3) = 11$ or $x = 11 - 6 = 5$.
   Thus, $x + y = 5 + 3 = 8$.

   ANSWER: $x + y = 8$

2. *Solution 1*

   We note that $9 + 9^2 + 9^3 + 9^4 = 9(1 + 9^1) + 9^3(1 + 9^1) = (9 + 9^3)(1 + 9) = 10(9 + 9^3)$.
   Therefore, $9 + 9^2 + 9^3 + 9^4$ is an integer that is divisible by 10, so its units digit is 0.

   Solution 2

   Calculating each term, $9^2 = 81$, $9^3 = 9^2 9^1 = 81(9) = 729$ and $9^4 = 9^3 9^1 = 729(9) = 6561$.
   Thus, $9 + 9^2 + 9^3 + 9^4 = 9 + 81 + 729 + 6561 = 90 + 7290 = 7380$.
   Therefore, the units digit of the integer equal to $9 + 9^2 + 9^3 + 9^4$ is 0.

   Solution 3

   Since $9^2 = 81$, its units digit is 1.
   Since $9^3 = 9^2 9^1$, then we can calculate the units digit of 9^3 by multiplying the units digits of 9^2 and 9^1 and finding the units digit of this product. (This is because the units digit of a product depends only on the units digits of the numbers that we are multiplying together.) Thus, the units digit of 9^3 is $1(9) = 9$.

Since $9^4 = 9^2 9^2$ and the units digit of 9^2 is 1, then the units digit of 9^4 is $1(1) = 1$, in a similar way.

We can now calculate the units digit of $9 + 9^2 + 9^3 + 9^4$ by adding the units digits of the four terms, and finding the units digit of this sum of units digits.

The sum of the units digits is $9 + 1 + 9 + 1 = 20$, so the units digit of the integer equal to $9 + 9^2 + 9^3 + 9^4$ is 0.

Answer: 0

3. Suppose the four positive integers are a, b, c, d.

   Since the average of the four positive integers is 8, then $\dfrac{a+b+c+d}{4} = 8$ or $a+b+c+d = 32$.

   We try to find the maximum possible value of d.

   We know that $d = 32 - a - b - c = 32 - (a + b + c)$.

   To make d as large as possible, we make $a + b + c$ as small as possible.

   Since a, b and c are different positive integers, then the smallest possible value of $a + b + c$ is $1 + 2 + 3$ or 6.

   Thus, the largest possible value of d is $32 - 6 = 26$.

Answer: 26

4. Consider $\triangle AED$ and $\triangle ACB$.

   These triangles have a common angle at A.

   Also, since DE is parallel to BC, then $\angle AED = \angle ACB$.

   Therefore, $\triangle AED$ is similar to $\triangle ACB$.

   Thus,

$$\begin{aligned} \frac{AE}{DE} &= \frac{AC}{BC} \\ \frac{x}{1} &= \frac{x + x^2 + 4}{6} \\ 6x &= x^2 + x + 4 \\ 0 &= x^2 - 5x + 4 \\ 0 &= (x - 1)(x - 4) \end{aligned}$$

   so $x = 1$ or $x = 4$.

   (We can check that each of these values of x actually gives a triangle.)

Answer: $x = 1$ or $x = 4$

5. Since p, q, r, s are four consecutive integers with $p < q < r < s$, then $r = s - 1$, $q = s - 2$, and $p = s - 3$.

Thus,

$$\begin{aligned} \tfrac{1}{2}p + \tfrac{1}{3}q + \tfrac{1}{4}r &= s \\ \tfrac{1}{2}(s-3) + \tfrac{1}{3}(s-2) + \tfrac{1}{4}(s-1) &= s \\ 6(s-3) + 4(s-2) + 3(s-1) &= 12s \qquad \text{(multiplying through by 12)} \\ 13s - 18 - 8 - 3 &= 12s \\ s &= 29\ . \end{aligned}$$

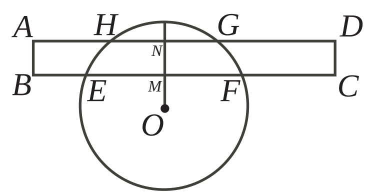
(Checking, $\frac{1}{2}(26) + \frac{1}{3}(27) + \frac{1}{4}(28) = 13 + 9 + 7 = 29$.)

Answer: $s = 29$

6. *Solution 1*

Let O be the centre of the circle and join O to the midpoints M and N of EF and HG, respectively.

Since O is the centre, then O, M and N are collinear and ON is perpendicular to both EF and HG.

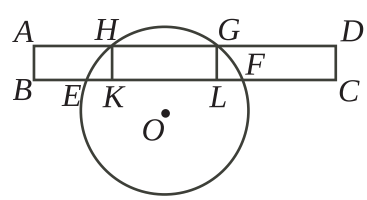


Since N is the midpoint of HG, then $HN = \frac{1}{2}HG = \frac{5}{2}$, so $AN = AH + HN = 4 + \frac{5}{2} = \frac{13}{2}$.
Since ON is perpendicular to AN and BM, and AN and BM are parallel, then $ANMB$ is a rectangle.
Therefore, $BM = AN = \frac{13}{2}$.
Thus,

$$EF = 2(EM) = 2(BM - BE) = 2(\tfrac{13}{2} - 3) = 2(\tfrac{7}{2}) = 7\ .$$

Solution 2

Let O be the centre of the circle.
Drop perpendiculars from G and H to K and L on EF, respectively.



Since $ABCD$ is a rectangle, then $AHKB$, $HGLK$ and $GDCL$ are rectangles.
We want the length of EF. We note that $EF = EK + KL + LF$.

Since $HGLK$ is a rectangle, then $KL = HG = 5$.
By symmetry, $EK = LF$.
Now $EK = BK - BE = AH - BE = 4 - 3 = 1$.
Thus, $EF = 1 + 5 + 1 = 7$.

Answer: $EF = 7$

7. We make a list of the possible paths that the star could take until it either reaches the top left corner or it moves off the grid. We use "L" to represent a move to the left and "U" to represent a move upwards. The possible paths are:

   LLL, LLUL, LLUU, LULL, LULU, LUUL, LUUU,
   UUU, UULU, UULL, ULUU, ULUL, ULLU, ULLL

   Of these paths, 6 of them reach the top left corner square, namely LLUU, LULU, LUUL, UULL, ULUL, and ULLU.
   Each time the star moves, the probability that it moves to the left is $\frac{1}{2}$ and the probability that it moves upwards is $\frac{1}{2}$. Thus, the probability that the star follows a particular path of length 3 is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and the probability that the star follows a particular path of length 4 is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. (We can note that we have 2 paths of length 3 and 12 paths of length 4, which is consistent with the probabilities above, because $2\left(\frac{1}{8}\right) + 12\left(\frac{1}{16}\right) = \frac{1}{4} + \frac{3}{4} = 1$.)
   Since there are 6 paths of length 4 that reach the top left corner, the probability that the star reaches this square is $6\left(\frac{1}{16}\right) = \frac{3}{8}$.

   Answer: $\frac{3}{8}$

8. Suppose that x is an integer that satisfies this equation.
   Then

$$\begin{aligned} x^3 - rx + r + 11 &= 0 \\ x^3 + 11 &= rx - r \\ r(x-1) &= x^3 + 11 \\ r &= \frac{x^3+11}{x-1} = \frac{x^3-1}{x-1} + \frac{12}{x-1} \\ r &= x^2 + x + 1 + \frac{12}{x-1} \end{aligned}$$

   We note that $x \neq 1$ (which we can confirm from the original equation, as if $x = 1$ we obtain $12 = 0$).

Thus, for r to be an integer, we need $\dfrac{12}{x-1}$ to be an integer, because $x^2 + x + 1$ is an integer already.
Hence, $x - 1$ must be a divisor of 12 and x must be positive (so $x - 1$ is non-negative). We make a table to enumerate the possible values of $x - 1$, and thus of x and r:

$x-1$	x	$r = x^2 + x + 1 + \dfrac{12}{x-1}$
12	13	184
6	7	59
4	5	34
3	4	25
2	3	19
1	2	19

Therefore, the sum of the possible values of r is

$$184 + 59 + 34 + 25 + 19 = 321$$

(Note that we only include 19 once in this sum, not two times.)

ANSWER: 321

Part B

1. (a) By the Pythagorean Theorem in $\triangle PSR$, we have

$$PR^2 = SR^2 + SP^2 = 9^2 + 12^2 = 81 + 144 = 225$$

so $PR = \sqrt{225} = 15$, since $PR > 0$.
By the Pythagorean Theorem in $\triangle PRQ$, we have

$$RQ^2 = PQ^2 - PR^2 = 25^2 - 15^2 = 625 - 225 = 400$$

so $RQ = \sqrt{400} = 20$, since $RQ > 0$.

(b) The area of figure $PQRS$ is the sum of the areas of $\triangle PSR$ and $\triangle PRQ$.
Since these triangles are right-angled at S and R, respectively, the area is

$$\tfrac{1}{2}(12)(9) + \tfrac{1}{2}(15)(20) = 54 + 150 = 204\ .$$

(c) *Solution 1*
We know all of the side-lengths of each of the two right-angled triangles $\triangle RSP$ and $\triangle PRQ$.
Therefore, we can calculate trigonometric ratios.
In particular, $\sin(\angle QPR) = \dfrac{RQ}{PQ} = \dfrac{20}{25} = \dfrac{4}{5}$ and $\sin(\angle PRS) = \dfrac{PS}{PR} = \dfrac{12}{15} = \dfrac{4}{5}$.
Since $\angle QPR$ and $\angle PRS$ are each acute and have equal sines, then $\angle QPR = \angle PRS$, as required.
(We could instead have calculated the cosines or tangents of these two angles.)

Solution 2
Since each of $\triangle RSP$ and $\triangle PRQ$ is right-angled (at S and at R, respectively) and $\dfrac{RS}{SP} = \dfrac{PR}{RQ} = \dfrac{3}{4}$, then $\triangle RSP$ is similar to $\triangle PQR$.
Thus, $\angle QPR = \angle SRP = \angle PRS$.

(d) Since $\angle QPR = \angle PRS$, then PQ is parallel to SR.
Since PS is perpendicular to SR and PQ is parallel to SR, then PS is also perpendicular to PQ, so $\angle SPQ = 90^\circ$.
Therefore, by the Pythagorean Theorem, $SQ^2 = SP^2 + PQ^2 = 12^2 + 25^2 = 144 + 625 = 769$, so $SQ = \sqrt{769}$ since $SQ > 0$.

2. (a) Since $(x+3)(x-6) = -14$, then $x^2 - 3x - 18 = -14$ or $x^2 - 3x - 4 = 0$.
Factoring, we obtain $(x-4)(x+1) = 0$, so $x = 4$ or $x = -1$.

(b) Let $u = 2^x$. Thus, $u^2 = (2^x)^2 = 2^{2x}$, so the equation becomes $u^2 - 3u - 4 = 0$.
By (a), $u = 4$ or $u = -1$ so $2^x = 4$ or $2^x = -1$.

The first gives $x = 2$ and the second does not have a solution since $2^x > 0$ for every real number x.
Thus, $x = 2$.

(c) Let $w = x^2 - 3x$. Thus, the equation can be rewritten as $w^2 = 4 - 3(-w)$ or $w^2 - 3w - 4 = 0$.
By (a), $w = 4$ or $w = -1$.
If $w = 4$, then $x^2 - 3x = 4$ or $x^2 - 3x - 4 = 0$ so $x = 4$ or $x = -1$ (by (a) again).
If $w = -1$, then $x^2 - 3x = -1$ or $x^2 - 3x + 1 = 0$. By the quadratic formula,

$$x = \frac{3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$$

Therefore, the solutions are $x = 4$, $x = -1$, and $x = \frac{3 \pm \sqrt{5}}{2}$.

3. (a) (i) Setting $m = n = 0$, the given equation becomes

$$a_0 + a_0 = \tfrac{1}{2}a_0 + \tfrac{1}{2}a_0$$

or $2a_0 = a_0$ or $a_0 = 0$.

(ii) Setting $m = 1$ and $n = 0$, the given equation becomes

$$a_1 + a_1 = \tfrac{1}{2}a_2 + \tfrac{1}{2}a_0$$

Since $a_0 = 0$ and $a_1 = 1$, we obtain $1 + 1 = \frac{1}{2}a_2 + \frac{1}{2}(0)$ or $\frac{1}{2}a_2 = 2$ or $a_2 = 4$.
One way to obtain a_3 in an equation by using values of m and n that are as small as possible is by setting $m = 2$ and $n = 1$. In this case, the equation becomes

$$a_1 + a_3 = \tfrac{1}{2}a_4 + \tfrac{1}{2}a_2 \qquad (*)$$

Since we already know the values of a_1 and a_2, then $(*)$ requires us to determine a_4 in order to determine a_3.
We can determine a_4 by setting $m = 2$ and $n = 0$, which gives

$$a_2 + a_2 = \tfrac{1}{2}a_4 + \tfrac{1}{2}a_0$$

from which we get $4 + 4 = \frac{1}{2}a_4 + \frac{1}{2}(0)$ or $\frac{1}{2}a_4 = 8$ or $a_4 = 16$.
Substituting back into $(*)$, we get $1 + a_3 = \frac{1}{2}(16) + \frac{1}{2}(4)$ or $1 + a_3 = 8 + 2$ or $a_3 = 9$.
Thus, if $a_1 = 1$, then $a_2 = 4$ and $a_3 = 9$.
(If we set $m = k$ and $n = 0$, the given equation would become $a_k + a_k = \frac{1}{2}a_{2k} + \frac{1}{2}a_0$. Since $a_0 = 0$, then $a_{2k} = 4a_k$, which is a more general statement than $a_2 = 4a_1$ and $a_4 = 4a_2$ that were discovered above.)

(b) *Solution 1*

Let M be a positive integer.

Setting $m = n = M$, the given equation becomes

$$b_0 + b_{2M} = b_{2M} + b_{2M}$$

or $b_0 + b_{2M} = 2b_{2M}$ or $b_0 = b_{2M}$.

This tells us that each even-numbered term in the sequence has the same value.

Next, set $m = M$ and $n = 0$. In this case, the given equation becomes

$$b_M + b_M = b_{2M} + b_0$$

Since $b_{2M} = b_0$, this equation becomes $2b_M = 2b_0$ or $b_M = b_0$.

Therefore, $b_M = b_0$ for every positive integer M.

Hence, all terms in the sequence have the same value, as required.

Solution 2

Let M be a positive integer.

Setting $m = n = M$, the given equation becomes

$$b_0 + b_{2M} = b_{2M} + b_{2M}$$

or $b_0 + b_{2M} = 2b_{2M}$ or $b_0 = b_{2M}$.

This tells us that each even-numbered term in the sequence has the same value.

Next, set $m = 1$ and $n = 0$. In this case, the given equation becomes

$$b_1 + b_1 = b_2 + b_0$$

Since $b_2 = b_0$, this equation becomes $2b_1 = 2b_0$ or $b_1 = b_0$.

Next, setting $m = M$ and $n = M - 1$, the equation becomes

$$b_1 + b_{2M-1} = b_{2M} + b_{2M-2}$$

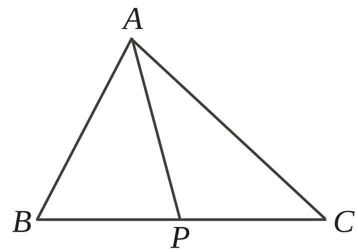
Since $b_1 = b_0$ (by the above) and $b_{2M} = b_{2M-2} = b_0$ (because these are even-numbered terms), this equation becomes $b_0 + b_{2M-1} = b_0 + b_0$ or $b_{2M-1} = b_0$.

Therefore, for any positive integer M, we have $b_{2M} = b_0$ and $b_{2M-1} = b_0$. In other words, every even-numbered term and every odd-numbered term is equal to b_0.

Hence, all terms in the sequence have the same value, as required.

4. Before solving the individual parts, we develop a formula that will allow us to calculate the lengths of the medians of any triangle, given its side lengths.

Consider $\triangle ABC$ with $AB = c$, $AC = b$ and $BC = a$. Let P, Q, R be the midpoints of BC, AC and AB, respectively, and m_a, m_b and m_c the lengths of the medians AP, BQ and CR, respectively.



We calculate the length of m_a first.
Consider $\triangle ABC$ and $\triangle ABP$.
By the cosine law in $\triangle ABC$,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos(\angle ABC) \\ b^2 &= c^2 + a^2 - 2ca\cos(\angle ABC) \\ 2ac\cos(\angle ABC) &= c^2 + a^2 - b^2 \\ \cos(\angle ABC) &= \frac{c^2 + a^2 - b^2}{2ac} \end{aligned}$$

By the cosine law in $\triangle ABP$, using $BP = \frac{1}{2}BC = \frac{1}{2}a$,

$$\begin{aligned} AP^2 &= AB^2 + BP^2 - 2(AB)(BP)\cos(\angle ABP) \\ (m_a)^2 &= c^2 + (\tfrac{1}{2}a)^2 - 2c(\tfrac{1}{2}a)\cos(\angle ABC) \\ (m_a)^2 &= c^2 + \tfrac{1}{4}a^2 - ac\left(\frac{c^2 + a^2 - b^2}{2ac}\right) \\ (m_a)^2 &= c^2 + \tfrac{1}{4}a^2 - \tfrac{1}{2}c^2 - \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 \\ (m_a)^2 &= \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2 \end{aligned}$$

(A second approach to calculating the length of m_a would be to consider $\triangle APB$ and $\triangle APC$, apply the cosine law in each focusing on vertex P, and use the fact that $\cos(\angle APB) = -\cos(\angle APC)$ since $\angle APB + \angle APC = 180^\circ$.)
So $(m_a)^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$.
Using a similar approach (or by relabelling the diagram), we can determine that

$$(m_b)^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}b^2$$

and

$$(m_c)^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}c^2$$

We are now ready to tackle parts (a) and (b).

(a) Suppose that $a = 7$, $b = 13$ and $c = 17$. From above,

$$(m_a)^2 = \tfrac{1}{2}(13^2) + \tfrac{1}{2}(17^2) - \tfrac{1}{4}(7^2) = \tfrac{169}{2} + \tfrac{289}{2} - \tfrac{49}{4} = 229 - \tfrac{49}{4} = \tfrac{867}{4}$$

and

$$(m_b)^2 = \tfrac{1}{2}(7^2) + \tfrac{1}{2}(17^2) - \tfrac{1}{4}(13^2) = \tfrac{49}{2} + \tfrac{289}{2} - \tfrac{169}{4} = 169 - \tfrac{169}{4} = \tfrac{507}{4}$$

and

$$(m_c)^2 = \tfrac{1}{2}(7^2) + \tfrac{1}{2}(13^2) - \tfrac{1}{4}(17^2) = \tfrac{49}{2} + \tfrac{169}{2} - \tfrac{289}{4} = 109 - \tfrac{289}{4} = \tfrac{147}{4}$$

Thus,

$$\begin{aligned} m_a &= \sqrt{\tfrac{867}{4}} = \sqrt{\tfrac{3(289)}{4}} = 17\tfrac{\sqrt{3}}{2} \\ m_b &= \sqrt{\tfrac{507}{4}} = \sqrt{\tfrac{3(169)}{4}} = 13\tfrac{\sqrt{3}}{2} \\ m_c &= \sqrt{\tfrac{147}{4}} = \sqrt{\tfrac{3(49)}{4}} = 7\tfrac{\sqrt{3}}{2} \end{aligned}$$

Since $a : b : c = 7 : 13 : 17$ and $m_c : m_b : m_a = 7 : 13 : 17$, then the lengths of the three medians m_c, m_b, m_a are in the same ratio as the lengths of the sides a, b, c, so the triangle formed with side lengths m_c, m_b, m_a is similar to the original triangle.
Therefore, the triangle with sides of length 7, 13 and 17 is automedian.
(Note that the fact that the lengths m_c, m_b, m_a are in the correct ratio (and in the same ratio as the side lengths of a triangle that already exists) also implies that a triangle can be formed with sides of lengths m_c, m_b, m_a.)

(b) Suppose that $\triangle ABC$ is automedian with $a < b < c$.
The first thing that we need to do is to determine the relative lengths of m_a, m_b and m_c.
In part (a), we saw that $m_c < m_b < m_a$. We verify that this is always the case whenever $a < b < c$.
First, we note that

$$(m_c)^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}c^2 < \tfrac{1}{2}a^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}b^2 = (m_b)^2$$

since $b < c$, so $m_c < m_b$ since $m_c > 0$ and $m_b > 0$.
Similarly,

$$(m_b)^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}b^2 < \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2 = (m_a)^2$$

since $a < b$, so $m_b < m_a$, which gives $m_c < m_b < m_a$.
Therefore, since the triangles with side lengths $a < b < c$ and $m_c < m_b < m_a$ are similar, then $m_c = ka$, $m_b = kb$ and $m_a = kc$ for some real number k.

Thus, we have

$$\begin{aligned}(m_c)^2 = k^2a^2 &= \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}c^2 \\ (m_b)^2 = k^2b^2 &= \tfrac{1}{2}a^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}b^2 \\ (m_a)^2 = k^2c^2 &= \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2\end{aligned}$$

Adding these three equations, we obtain

$$k^2(a^2 + b^2 + c^2) = \tfrac{3}{4}a^2 + \tfrac{3}{4}b^2 + \tfrac{3}{4}c^2$$

and so $k^2 = \frac{3}{4}$ or $k = \frac{\sqrt{3}}{2}$ since $k > 0$.

We can then substitute this value of k^2 into any of the three equations for the lengths of the medians. We substitute into the equation for m_c, obtaining $\frac{3}{4}a^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2$ or $\frac{1}{4}a^2 + \frac{1}{4}c^2 = \frac{1}{2}b^2$ or $a^2 + c^2 = 2b^2$, as required.

(It is interesting to note that, even if a triangle is not automedian, its three medians will always form a triangle, and a triangle whose area is $\frac{3}{4}$ of the area of the original triangle. This is consistent with the ratio of similarity found here.)

(c) We approach this part by trying to find a few triples of positive integers (a, b, c) with $a < b < c$ that satisfy $a^2 + c^2 = 2b^2$ then by finding a pattern to write down a general form for an infinite family that appears to satisfy $a^2 + c^2 = 2b^2$. We will then need to prove that this infinite family works. We do not yet know that if $a^2 + c^2 = 2b^2$, then the triangle is automedian, but we will prove this too. (We proved the converse in (b).)

We are looking for triples (a, b, c) that satisfy $a^2+c^2 = 2b^2$, or equivalently, $c^2-b^2 = b^2-a^2$. (In other words, a^2, b^2, c^2 form an arithmetic sequence.)

After some trial and error, we can find the triples $(1, 5, 7)$, $(7, 13, 17)$, $(17, 25, 31)$ that satisfy this equation. Note that the largest number in each triple becomes the smallest in the next triple and the differences between consecutive numbers in each triple are two more than in the previous triple. (Note that the triple $(1, 5, 7)$ is a triple that satisfies $a^2 + c^2 = 2b^2$, but a triangle cannot be formed with sides of these lengths. We keep this triple in our list in any event as it helps us find a pattern.)

We try to write down an infinite family (a_n, b_n, c_n) that follows from these three triples.

If we label the first triple with $n = 1$, the second with $n = 2$ and the third with $n = 3$, we might see that $7 = 2(2^2) - 1$, $17 = 2(3^2) - 1$ and $31 = 2(4^2) - 1$.

Thus, we try $c_n = 2(n + 1)^2 - 1 = 2n^2 + 4n + 1$.

The difference between b and c in the first is 2, in the second is 4 and in the third is 6, so we try $b_n = c_n - 2n = 2n^2 + 2n + 1$.

The difference between a and b in the first is 4, in the second is 6 and in the third is 8, so we try $a_n = b_n - 2(n + 1) = 2n^2 - 1$.

Thus, we try the triples $(a_n, b_n, c_n) = (2n^2 - 1, 2n^2 + 2n + 1, 2n^2 + 4n + 1)$ for n a positive integer. (The three triples found above do fit this formula.)

Since n is a positive integer, these are triples of positive integers. We need to check that

- these triples actually form triangles,
- the triangles formed are automedian, and
- no two of the triangles formed are similar.

Step 1: If $n \geq 2$, then (a_n, b_n, c_n) are the side lengths of a triangle

For this to be the case, the three lengths must obey the triangle inequality. That is, the sum of any two of the lengths must be greater than the third length. Since $a_n < b_n < c_n$, this means that we only need to check if $a_n + b_n > c_n$ (since $a_n + c_n > b_n$ and $b_n + c_n > a_n$ are automatic).

Here, $(2n^2 - 1) + (2n^2 + 2n + 1) > (2n^2 + 4n + 1)$ is equivalent to $4n^2 + 2n > 2n^2 + 4n + 1$ which is equivalent to $2n^2 - 2n > 1$ which is equivalent to $2n(n-1) > 1$, which is true when $n \geq 2$, as the left side will be at least $2(2)(1) = 4$.

Step 2: If $n \geq 2$, then (a_n, b_n, c_n) are the side lengths of an automedian triangle

We do this in two steps – we show that $(a_n)^2 + (c_n)^2 = 2(b_n)^2$ and then show that if a triangle has $a^2 + c^2 = 2b^2$, then it is automedian.

We show that $(a_n)^2 + (c_n)^2 = 2(b_n)^2$ by showing that $(c_n)^2 - (b_n)^2 = (b_n)^2 - (a_n)^2$.

We have

$$(c_n)^2 - (b_n)^2 = (c_n + b_n)(c_n - b_n) = (4n^2 + 6n + 2)(2n) = 4n(2n^2 + 3n + 1) = 4n(n+1)(2n+1)$$

and

$$(b_n)^2 - (a_n)^2 = (b_n + a_n)(b_n - a_n) = (4n^2 + 2n)(2n + 2) = 4n(2n+1)(n+1)$$

as required.

Next, suppose that a triangle with side lengths (a, b, c) satisfies $a^2 + c^2 = 2b^2$.

Then, from our formulae above,

$$\begin{aligned}
(m_c)^2 &= \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}c^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}(2b^2 - a^2) = \tfrac{3}{4}a^2 \\
(m_b)^2 &= \tfrac{1}{2}a^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}b^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}(2b^2 - a^2) - \tfrac{1}{4}b^2 = \tfrac{3}{4}b^2 \\
(m_a)^2 &= \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2 = \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}(2b^2 - c^2) = \tfrac{3}{4}c^2
\end{aligned}$$

so $m_c = \frac{\sqrt{3}}{2}a$, $m_b = \frac{\sqrt{3}}{2}b$, and $m_a = \frac{\sqrt{3}}{2}c$.

Thus, if $a^2 + c^2 = 2b^2$, then the triangle is automedian.

Therefore, if $n \geq 2$, then (a_n, b_n, c_n) are the side lengths of an automedian triangle.

Step 3: No two (a_n, b_n, c_n) with $n \geq 2$ form similar triangles

Suppose that (a_n, b_n, c_n) and (a_m, b_m, c_m) are two similar triangles, with $a_m = da_n$, $b_m = db_n$ and $c_m = dc_n$ for some real number d.

$$\begin{aligned} 2m^2 - 1 &= d(2n^2 - 1) \\ 2m^2 + 2m + 1 &= d(2n^2 + 2n + 1) \\ 2m^2 + 4m + 1 &= d(2n^2 + 4n + 1) \end{aligned}$$

Subtracting the second of these equations from the third, we obtain $2m = 2dn$ or $m = dn$.
Subtracting the first of these equations from the second, we obtain $2m + 2 = d(2n + 2)$ or $m + 1 = d(n + 1)$.
Since $m = dn$, we obtain $dn + 1 = dn + d$ or $d = 1$.
Since $d = 1$, then $n = m$. Therefore, if $n \neq m$, the triangles with side lengths (a_m, b_m, c_m) and (a_n, b_n, c_n) are not similar.

Therefore, the infinite family $(a_n, b_n, c_n) = (2n^2 - 1, 2n^2 + 2n + 1, 2n^2 + 4n + 1)$ with n a positive integer and $n \geq 2$ is an infinite family of automedian triangles with integer side lengths, no two of which are similar.
(There are other possible infinite families that work too.)